Lecture 19 | 3 March 2010

Hwks: "REVIEW 1" is not another HW assignment
(just for you to review for the final!)

Partial fractions (Sec 8.4)

How to integrate complicated rational functions $\frac{P(x)}{Q(x)}$
[P, Q polynomials]

Ex. $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

Factor the denominator:

$$2x^3+3x^2-2x = x(2x^2+3x-2)$$
$$= x(2x-1)(x+2)$$

Then set $\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$

To find A, B, C: multiply both sides by the denominator $x(2x-1)(x+2)$

$$x^2+2x-1 = A(2x-1)(x+2) + B(x)(x+2) + C(x)(2x-1)$$
$$= A(2x^2+3x-2) + B(x^2+2x) + C(2x^2-x)$$
$$= (2A+B+2C)x^2 + (3A+2B-C)x - 2A\,(1)$$

Equate the coefficients:
$$1 = 2A+B+2C$$
$$2 = 3A+2B-C$$
$$-1 = -2A$$

Solve these eq: $A = \frac{1}{2}$
$B = \frac{1}{5}$
$C = -\frac{1}{10}$

$1 = 1 + B + 2C$
$2 = \frac{3}{2} + 2B - C$

$B + 2C = 0$
$B = -2C$

$2 = \frac{3}{2} - 4C - C$
$\frac{1}{2} = -5C$
$C = -\frac{1}{10}$

So $\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2x-1} - \frac{1}{10}\frac{1}{x+2}$

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx = \int \left(\frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2x-1} - \frac{1}{10}\frac{1}{x+2}\right)dx$$
$$= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + K$$

What if the denominator doesn't factor completely (into linear factors)?

Ex $\displaystyle\int \frac{2x^2-x+4}{x^3+4x}\,dx$

Factor: $x^3+4x = x(x^2+4)$

Write $\displaystyle\frac{2x^2-x+4}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$

To find A, B, C: mult. both sides by $x(x^2+4)$

$$\begin{aligned} 2x^2-x+4 &= A(x^2+4)+(Bx+C)(x) \\ &= Ax^2+4A+Bx^2+Cx \\ &= (A+B)x^2+Cx+4A \end{aligned}$$

Equate coeff:

$$\left.\begin{aligned} 2&=A+B \\ -1&=C \\ 4&=4A \end{aligned}\right\} \longrightarrow \begin{aligned} A&=1 \\ B&=1 \\ C&=-1 \end{aligned}$$

So $\displaystyle\int = \int \frac{1}{x} + \frac{x-1}{x^2+4}\,dx$

$$= \int \underbrace{\frac{1}{x}}_{\ln|x|} + \underbrace{\frac{x}{x^2+4}}_{\text{use } u=x^2+4,\ \text{get } \frac{1}{2}\ln(x^2+4)} - \underbrace{\frac{1}{x^2+4}}_{\text{use } u=\frac{x}{2},\ \text{get } -\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)}\,dx$$

$$= \underline{\ln|x| + \tfrac{1}{2}\ln(x^2+4) - \tfrac{1}{2}\tan^{-1}\left(\tfrac{x}{2}\right) + K}$$

What if the degree of the numerator $\geq$ the degree of the denominator?

Ex $\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx$

Divide first:

$$\begin{array}{r} x+1 \\ x^2-x-6 \overline{) x^3+0x^2-4x-10} \\ \underline{x^3-x^2-6x} \\ x^2+2x-10 \\ \underline{x^2-x-6} \\ 3x-4 \end{array}$$

So $\frac{x^3-4x-10}{x^2-x-6} = x+1+\frac{3x-4}{x^2-x-6}$

So $\int \frac{x^3-4x-10}{x^2-x-6} = \int x+1+\frac{3x-4}{x^2-x-6}\,dx$

Factor: $x^2-x-6=(x-3)(x+2)$

$$\frac{3x-4}{x^2-x-6} = \frac{A}{x-3}+\frac{B}{x+2}$$

$$3x-4 = A(x+2)+B(x-3)$$

$$3x-4 = (A+B)x+(2A-3B)$$

$$\left.\begin{array}{r} 3=A+B \\ -4=2A-3B \end{array}\right] \Rightarrow A=1, B=2$$

So $\int = \int_0^1 x+1+\frac{1}{x-3}+\frac{2}{x+2}\,dx = \cdots = \underline{\underline{\frac{3}{2}+\ln\frac{3}{2}}}$

What if some factor appears <u>more than once</u> in the denom?

<u>Ex</u> $$\int \frac{1}{x^3+2x^2+x}\,dx$$

Factor: $$x^3+2x^2+x = x(x^2+2x+1)$$
$$= x(x+1)^2$$

Write $$\frac{1}{x^3+2x^2+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \leftarrow !$$

Mult. by $x(x+1)^2$ both sides:

$$1 = A(x+1)^2 + Bx(x+1) + Cx$$
$$= A(x^2+2x+1) + B(x^2+x) + Cx$$
$$= (A+B)x^2 + (2A+B+C)x + A$$

$$\Rightarrow \quad \begin{aligned} A+B&=0 \\ 2A+B+C&=0 \\ A&=1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} A&=1 \\ B&=-1 \\ C&=-1 \end{aligned}$$