Lecture 1

28 Aug 2014

Welcome! M408M, Fall 2014. Multivariate Calculus.

Course web page: www.ma.utexas.edu/users/neitzke/teaching/408M

— contains first day handout, will contain lecture slides posted after each class

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Lectures TuTh 11:00–12:30, CPE 2.214 (11:00–12:15 ± 2 min)

Discussion MW 8:00–9:00, UTC 1.116

or MW 2:00–3:00, CPE 2.220

Office hours Instructor: MW 11:00–12:00, RLM 9.134

TA: M 3:00–4:30, RLM 12.146

F 1:00–2:30, RLM 12.146

Homework via QUEST at quest.cns.utexas.edu — 15 %

Due at 3am each Tue morning — can submit as you go

Lowest 2 dropped from grade

Working together strongly recommended!

2 midterm exams (in class) — 50 %

Oct 2, Nov 4

Can replace lowest midterm grade with final grade

Final exam — 35 %

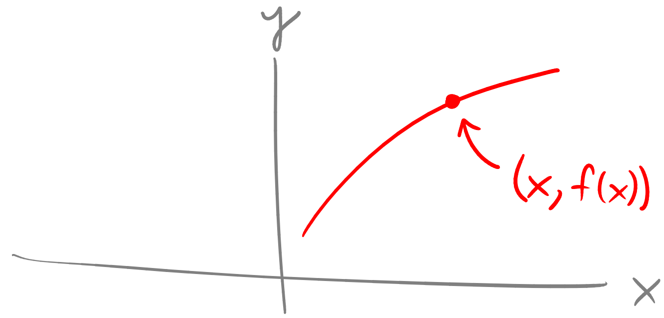
Many resources available:

- Office hours
- Lecture slides
- Textbook
- CalcLab (M-F 2-7, Painter 5.42)
- Sanger Center (Jester A115)
- Your fellow students!

Parameterized Curves (Ch 10.1)

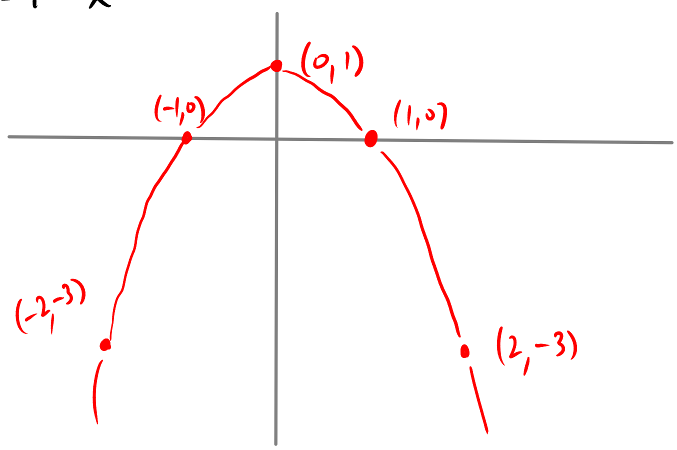
Until now, you've mostly considered $y = f(x)$

Graph of the function:



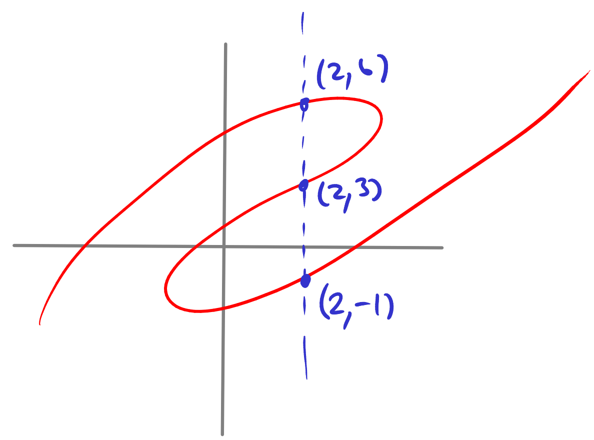
e.g. $f(x) = 1 - x^2$ $\qquad$ $y = 1 - x^2$

x	y
0	1
1	0
-1	0
2	-3
-2	-3



But, we can't represent every curve this way!

(Vertical line test)



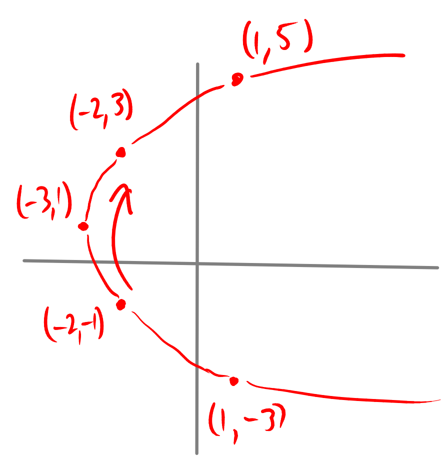
Instead, try writing:

$$x = f(t)$$
$$y = g(t)$$

Ex Say we put

$$x = t^2 - 3$$
$$y = 2t + 1$$

t	x	y
2	1	5
1	-2	3
0	-3	1
-1	-2	-1
-2	1	-3



Looks like parabola! To see it is parabola, eliminate t:

$$y = 2t+1$$

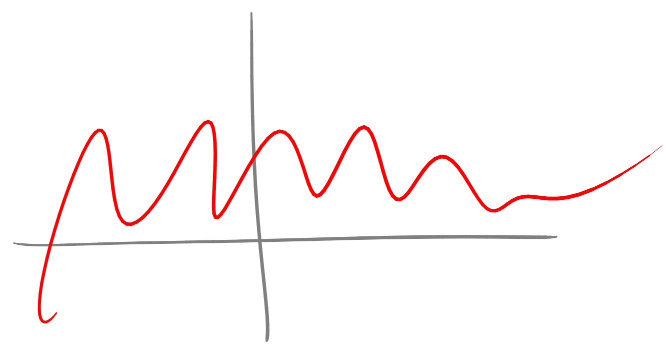
$$t = \tfrac{1}{2}(y-1) \qquad x = t^2 - 3$$

$$= \left(\tfrac{1}{2}(y-1)\right)^2 - 3$$

$$x = \tfrac{1}{4}y^2 - y - \tfrac{11}{4} \qquad \text{parabola}$$

Q: Say $y = f(x)$

$$y = \frac{1}{\sin(x^3-4)} + \tan(x)$$



Can this be written in parametric form?

Yes: take

$$x = t$$

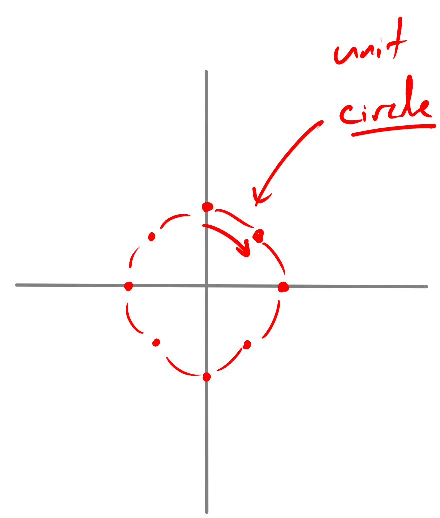
$$y = \frac{1}{\sin(t^3-4)} + \tan(t)$$

Ex (Most famous example)

$$x = \sin t$$

$$y = \cos t$$

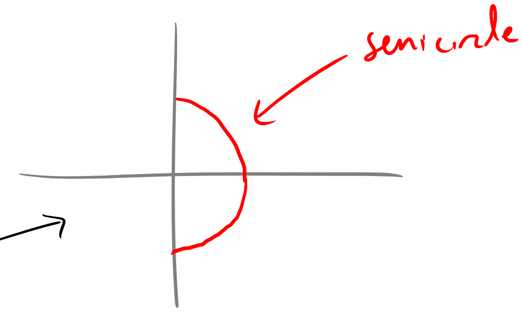
t	x	y
0	0	1
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/2$	1	0
π	0	-1
$3\pi/2$	-1	0
2π	0	1



$$x^2 + y^2 = \sin^2 t + \cos^2 t = 1$$

If we restrict t to $0 \le t \le \pi$

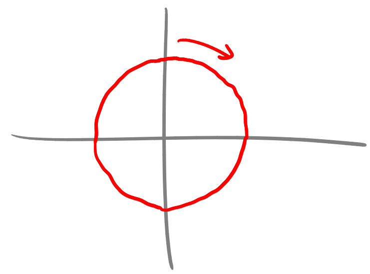
then get



Ex $x = \sin(3t)$

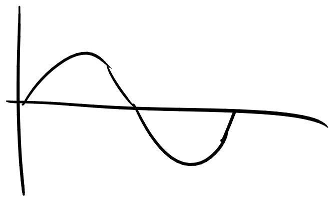
$y = \cos(3t)$

$x^2 + y^2 = \sin^2(3t) + \cos^2(3t) = 1$ so still unit circle

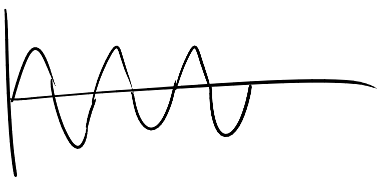


but we "go around $3\times$ faster"

$\sin(t)$



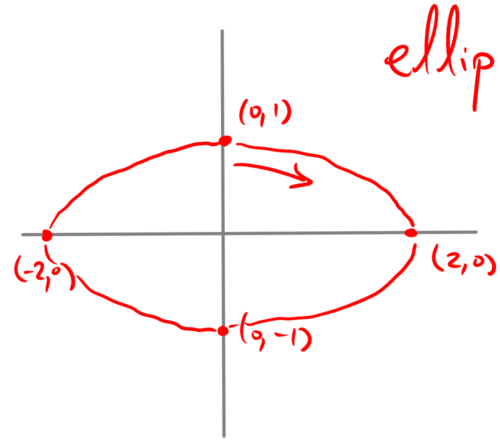
$\sin(3t)$



Ex $x = 2\sin t$

$y = \cos t$

t	x	y
0	0	1
$\pi/4$	$\sqrt{2}$	$\sqrt{2}/2$
$\pi/2$	2	0
π	0	-1
$3\pi/2$	-2	0
2π	0	1



ellipse

$x^2 + y^2 = 4\sin^2 t + \cos^2 t$

$\frac{x^2}{4} + y^2 = \sin^2 t + \cos^2 t = 1$

$\frac{x^2}{4} + y^2 = 1$

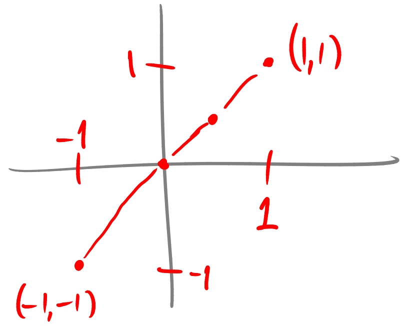
eq. of ellipse

w/ minor axis 1
major axis 2

$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a=2, b=1\right)$

Ex $x = \sin t$
$y = \sin t$

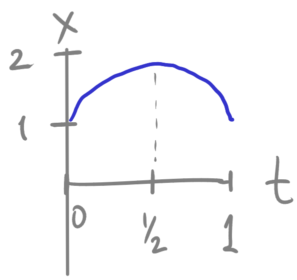
t	x	y
0	0	0
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/2$	1	1
π	0	0
$3\pi/2$	-1	-1
2π	0	0

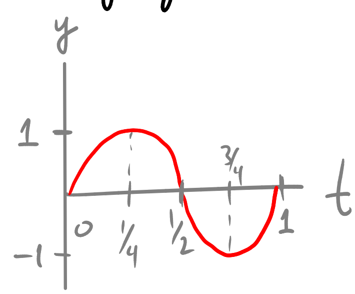


$y = x$ but not <u>every</u> point on the line is included: only those with $-1 \leq x \leq 1$

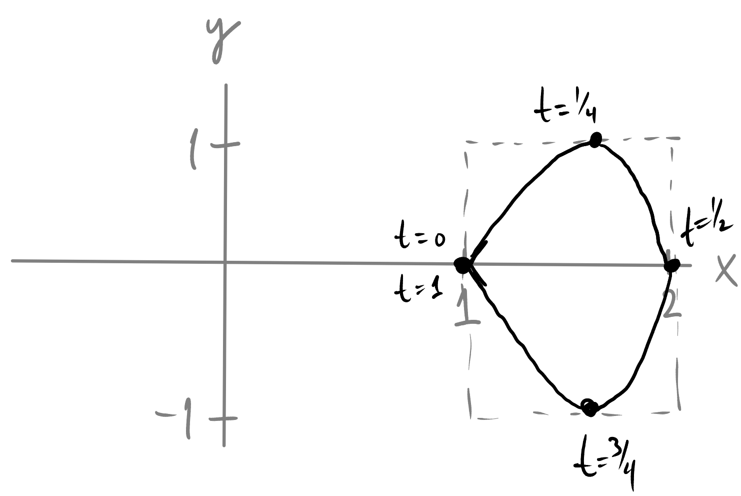
Oscillate back and forth between $(-1,-1)$ and $(1,1)$.

Ex Sketch the curve given by $x = x(t)$ $y = y(t)$





Answer:

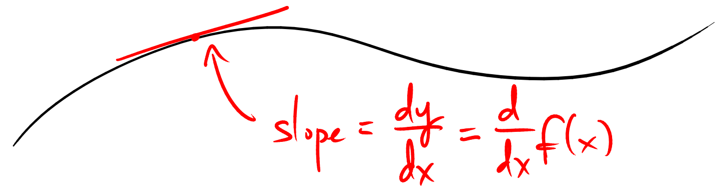


Ex Try $x = \sin 2t$
$y = \sin t$!

Calculus with Parameterized Curves (Ch 10.2)

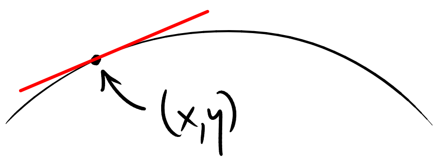
Finding tangents

Recall:



$y = f(x)$

If we have a parameterized curve:



$x = f(t)$
$y = g(t)$

What is the slope of the tangent line?

$$\text{slope} = \frac{(dy/dt)}{(dx/dt)}$$ as long as this is not $\frac{0}{0}$!

Ex Say $x(t) = 3t^2 + 1$, $y(t) = 7/t$. Find slope of tangent line to this curve at $(x,y) = (4,7)$.

$dy/dt = -\frac{7}{t^2}$

$dx/dt = 6t$

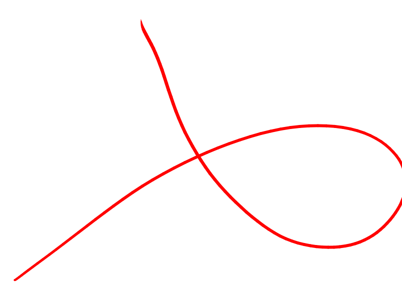
and we evaluate at $t=1$: [because $y=7 \Rightarrow 7 = \frac{7}{t} \Rightarrow t=1$]

$dy/dt = -7$

$dx/dt = 6$

$$\text{slope} = \frac{dy/dt}{dx/dt} = \frac{-7}{6}$$

[Rk For a more complicated parameterized curve, knowing (x,y) might not determine t!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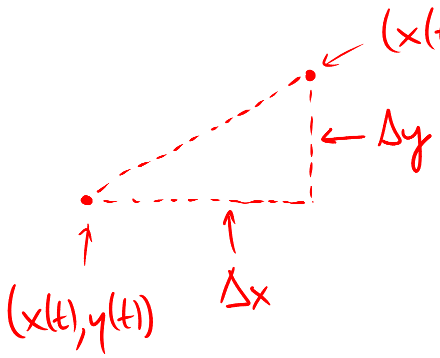
Why does the formula $\text{slope} = \dfrac{dy/dt}{dx/dt}$ work?

(Mnemonic: "cancel the dt's" to get dy/dx)

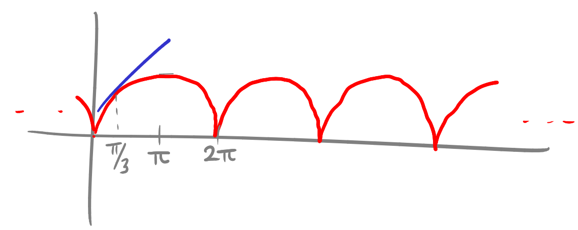
Imagine moving from t to $t+\Delta t$, for Δt very small:



$$\text{slope} \approx \frac{\Delta y}{\Delta x} \approx \frac{dy/dt \cdot \Delta t}{dx/dt \cdot \Delta t} = \frac{dy/dt}{dx/dt}$$

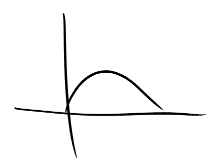
Ex Cycloid

$$x = t - \sin t$$
$$y = 1 - \cos t$$



a) find slope of tangent line at $t = \pi/3$

$$\text{slope} = \frac{dy/dt}{dx/dt} = \frac{\sin t}{1-\cos t} = \frac{\sqrt{3}/2}{1-\frac{1}{2}} = \sqrt{3}$$



b) find the values of t when cycloid has horizontal or vertical tangents.

Horizontal tangents: slope $= 0$ ie $\dfrac{dy}{dt} = 0$ and $\dfrac{dx}{dt} \neq 0$ ie $\sin t = 0$, $\cos t \neq 1$ ie $t = (2n+1)\pi$, n any integer

Vertical tangents: slope $= \infty$ ie $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} \neq 0$ ie $\cos t = 1$, $\sin t \neq 0$ — this never happens

But, we should also consider what happens at places where both $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$

This happens at $t = 2n\pi$, n any integer.

In this case, the slope formula $\frac{dy/dt}{dx/dt}$ gives $\frac{0}{0}$.

We get the correct slope in this case by L'Hospital Rule:

$$\lim_{t \to 0} \frac{\sin t}{1 - \cos t} = \lim_{t \to 0} \frac{\cos t}{\sin t} = +\infty$$

Thus we do get vertical tangents at these points!

So altogether: horizontal tangents at $t = (2n+1)\pi$

vertical tangents at $t = 2n\pi$