Lecture 3

4 Sep 2014

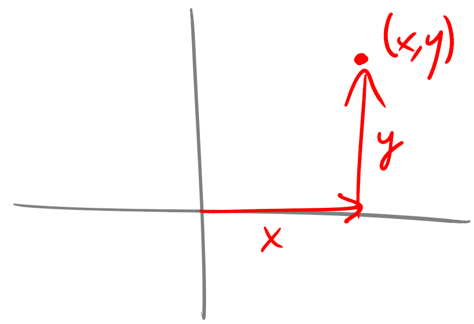
Note: ① After the HW deadline, QUEST allows you to see worked solutions to the problems!

② Calc Lab is now open! 2-7pm M-F
Painter 5.42

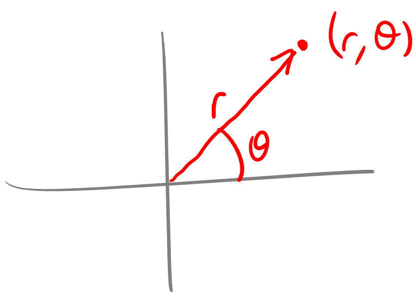
Last time: calculus w/ parameterized curves

Polar coordinates (Ch 10.3)

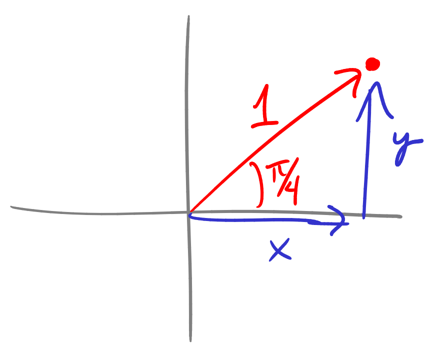
We usually use coords (x,y) in the plane.



An alternative coordinate system: $(r,\theta) \leftarrow$ polar coordinates



Ex The point with polar coordinates $(r,\theta) = (1, \pi/4)$ has x-y coordinates $(x,y) = ?$



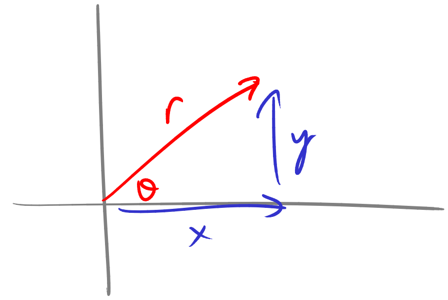
$$\sin\frac{\pi}{4} = \frac{y}{1} \qquad \cos\frac{\pi}{4} = \frac{x}{1}$$

$$\frac{\sqrt{2}}{2} = y \qquad \frac{\sqrt{2}}{2} = x$$

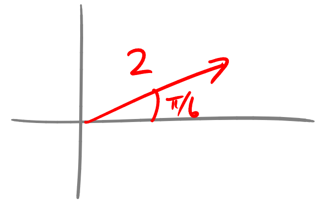
so $(x,y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

In general, to convert from polar to (x,y):

$$x = r\cos\theta$$
$$y = r\sin\theta$$

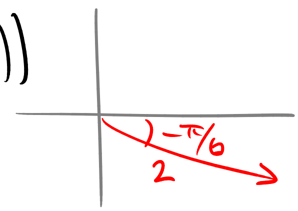


Ex $(r,\theta) = (2, \pi/6) \longrightarrow (x,y) = \left(2\cos\frac{\pi}{6},\ 2\sin\frac{\pi}{6}\right)$
$= (\sqrt{3},\ 1)$

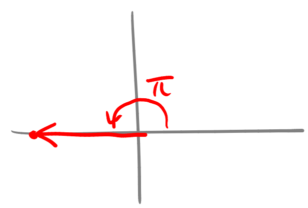


The formulas work even for (x,y) not in the 1st quadrant:

Ex $(r,\theta) = (2, -\pi/6) \rightarrow (x,y) = \left(2\cos\left(-\frac{\pi}{6}\right),\ 2\sin\left(-\frac{\pi}{6}\right)\right)$
$= (\sqrt{3}, -1)$



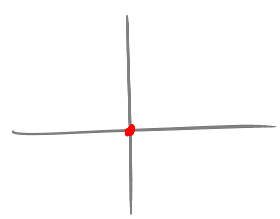
Ex $(r,\theta) = (6, \pi) \rightarrow (x,y) = (6\cos(\pi),\ 6\sin(\pi))$
$= (-6, 0)$



Could also describe the same point by
$(r,\theta) = (6, -\pi)$ (or $(r,\theta) = (6, 3\pi)$ or even $(r,\theta) = (6, \pi + 2n\pi)$ for any n)

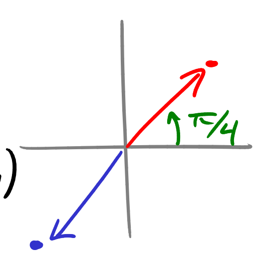
So, different (r,θ) might describe the same point!

Ex $(r,\theta) = (0, \text{anything}) \longrightarrow (x,y) = (0,0)$



Can also allow $r < 0$.

Ex $(r,\theta) = (+1, \pi/4) \longrightarrow (x,y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$(r,\theta) = (-1, \pi/4) \longrightarrow (x,y) = (-1\cdot\cos\pi/4,\ -1\cdot\sin\pi/4)$
$= \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$



Converting back:

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \longrightarrow x^2+y^2 = r^2(\sin^2\theta+\cos^2\theta) = r^2$$

so $r = \sqrt{x^2+y^2}$ (if $r>0$)

$$\downarrow$$

$$\frac{y}{x} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

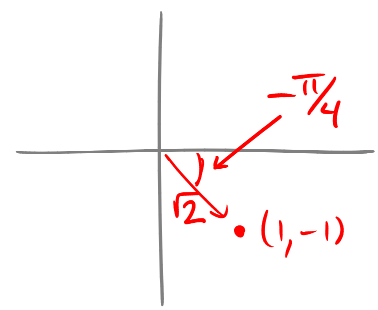
so $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ (if $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, ie if $x>0$)

$\theta = \tan^{-1}\left(\frac{y}{x}\right)+\pi$ (if not)

Ex $(x,y) = (1,-1)$: $r = \sqrt{1^2+(-1)^2} = \sqrt{2}$

$$\theta = \tan^{-1}\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$$

(here we don't have to shift θ by π, because (x,y) is in the right ½-plane, i.e. $x>0$)

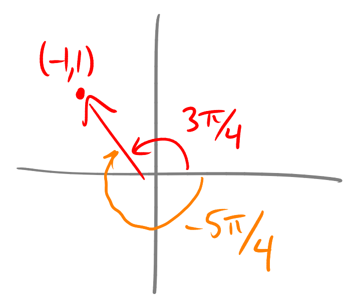


$(x,y) = (-1,1)$: $r = \sqrt{(-1)^2+1^2} = \sqrt{2}$

$$\theta = \tan^{-1}\left(\frac{1}{-1}\right)+\pi = -\frac{\pi}{4}+\pi$$

$$= \frac{3\pi}{4}$$

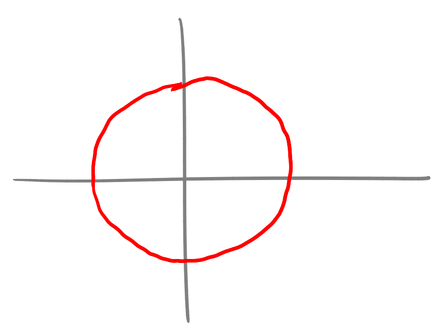
(we shift by π because (x,y) is in left ½-plane, i.e. $x<0$)



Polar curves

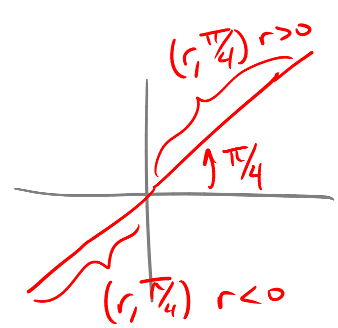
Ex $r=3$

$\longrightarrow$ circle of radius 3



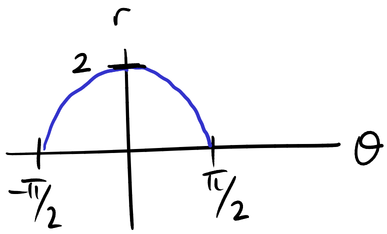
$x = 3\cos\theta$
$y = 3\sin\theta$

Ex $\theta = \pi/4$

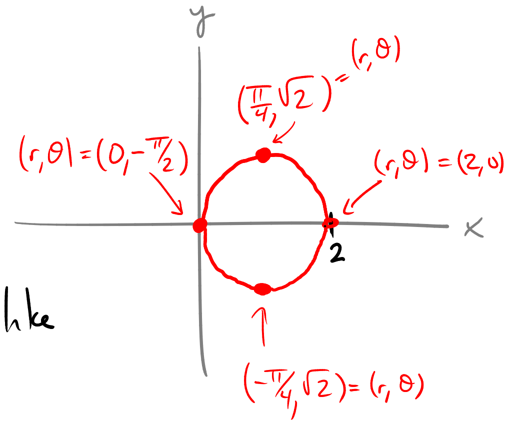


Ex $r = 2\cos\theta \quad \left[-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right]$ $\qquad r = r(\theta)$

θ	r
$-\pi/2$	0
$-\pi/4$	$\sqrt{2}$
0	2
$\pi/4$	$\sqrt{2}$
$\pi/2$	0



$\longrightarrow$ looks roughly like



In fact, this polar curve is a circle! To see this, convert to (x,y):

$$r = 2\cos\theta = 2\left(\frac{x}{r}\right)$$

$$r = \frac{2x}{r}$$

$$r^2 = 2x$$

$$x^2 + y^2 = 2x$$

Looks like a circle except for the extra $2x$.
Complete square to absorb that:

$$x^2 - 2x + y^2 = 0$$

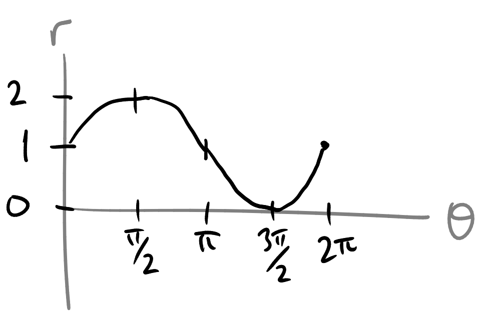
[In general: $x^2 + ax$
$= x^2 + ax + \frac{a^2}{4} - \frac{a^2}{4}$
$= (x + \frac{a}{2})^2 - \frac{a^2}{4}$]

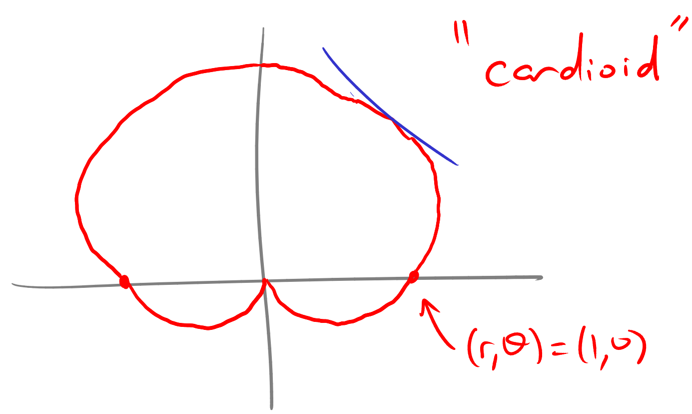
$$x^2 - 2x + 1 + y^2 = 1$$

$$(x-1)^2 + y^2 = 1$$

$\longrightarrow$ circle with center $(x,y) = (1,0)$
radius 1

Ex $r = 1 + \sin\theta$





To get a more precise picture, want to be able to find slopes — e.g. find <u>horizontal</u>, <u>vertical tangents</u>.

Treat $r = r(\theta)$ as a parameterized curve:

$$x = r(\theta)\cos\theta$$
$$y = r(\theta)\sin\theta$$

parameterized by θ

then use $\text{slope} = \dfrac{dy/d\theta}{dx/d\theta}$

<u>Ex</u> Find the slope of the tangent line to the cardioid at $\theta = \pi/3$.

Cardioid: $r = 1 + \sin\theta$

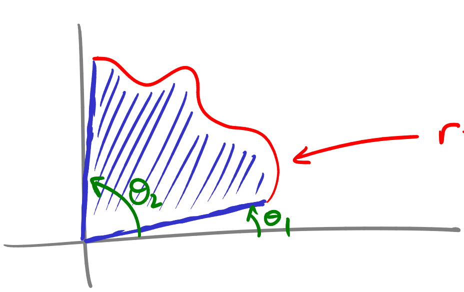
$$x = (1+\sin\theta)\cos\theta$$
$$y = (1+\sin\theta)\sin\theta$$

$$\text{slope} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}\left[\sin\theta + \sin^2\theta\right]}{\frac{d}{d\theta}\left[\cos\theta + \sin\theta\cos\theta\right]}$$

evaluated at $\theta = \pi/3$

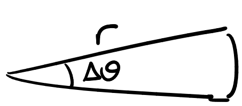
$$= \cdots = -1$$

Areas + Lengths in Polar Coordinates



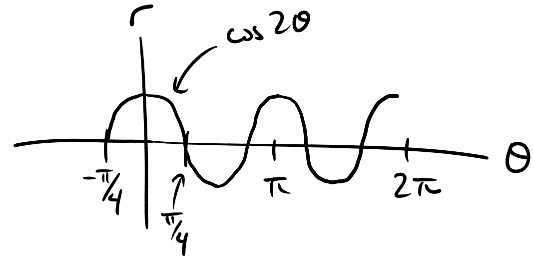
How to compute the <u>area</u> of the shaded region?

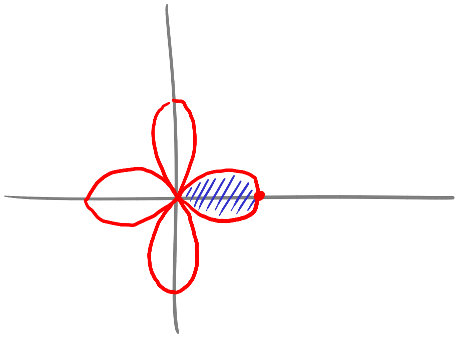
$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r(\theta)^2 \, d\theta$$

[Why? 

has area $= \dfrac{\Delta\theta}{2\pi}\pi r^2 = \frac{1}{2} r^2 \Delta\theta$]

Ex $r = \cos 2\theta$





Find the area of the shaded region.

$$A = \int_{-\pi/4}^{\pi/4} \frac{1}{2} r(\theta)^2 \, d\theta$$

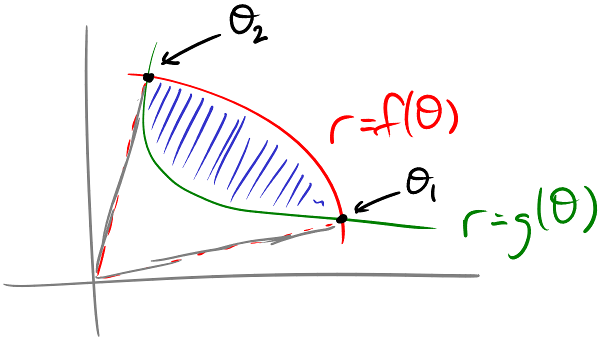
$$= \int_{-\pi/4}^{\pi/4} \frac{1}{2} \cos^2 2\theta \, d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \frac{1}{4}(1+\cos 4\theta) \, d\theta$$

$$= \frac{\theta}{4} + \frac{\sin 4\theta}{16} \Big]_{-\pi/4}^{\pi/4}$$

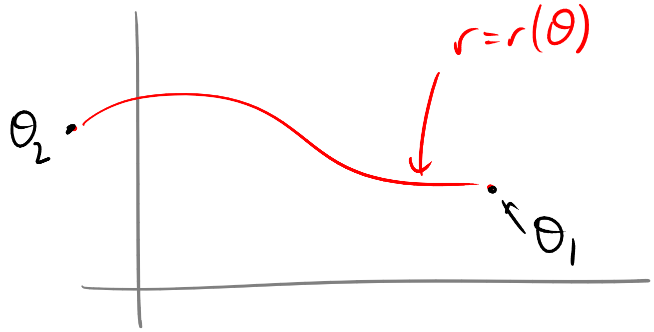
$$= \frac{1}{4}\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \frac{\pi}{8}$$

Area between polar curves:



$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} \left(f(\theta)^2 - g(\theta)^2\right) d\theta$$

Lengths of Polar Curves



Again, do it as for parameterized curve:

$$L = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$x = r(\theta)\cos\theta$$
$$y = r(\theta)\sin\theta$$

$$= \quad \cdots$$

$$= \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad \longleftarrow \quad \text{just use this!}$$