Lecture 5

11 Sep 2014

Last time: 3d coordinate systems + vectors

Ex What is the locus $x^2+y^2+z^2-2x-4y-8z=15$ in 3d?

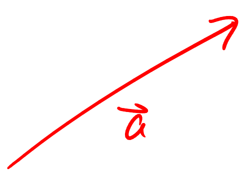
Complete square:

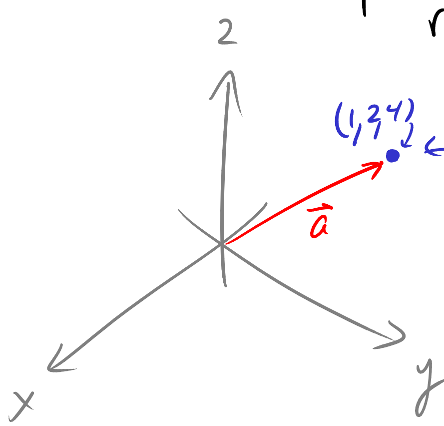
$$x^2-2x = (x-1)^2-1$$
$$y^2-4y = (y-2)^2-4$$
$$z^2-8z = (z-4)^2-16$$

$$(x-1)^2-1+(y-2)^2-4+(z-4)^2-16=15$$
$$(x-1)^2+(y-2)^2+(z-4)^2=15+1+4+16$$
$$(x-1)^2+(y-2)^2+(z-4)^2=36$$

sphere centered at $(1,2,4)$
radius 6

Vectors





coordinates of this point are the components of $\vec{a}$.

$\vec{a}=\langle 1,2,4\rangle$

$a_1=1 \quad a_2=2 \quad a_3=4$

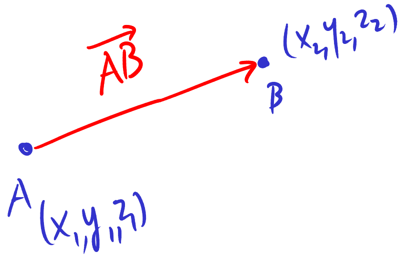
or $a_x=1 \quad a_y=2 \quad a_z=4$

Length of vector: $\|\vec{a}\|=\sqrt{a_1^2+a_2^2+a_3^2}$ for vector in 3d

$\|\vec{a}\|=\sqrt{a_1^2+a_2^2}$ " " " 2d

Ex If $\vec{a}=\langle 1,2,-7\rangle$ then $\|\vec{a}\|=\sqrt{1^2+2^2+(-7)^2}=\sqrt{54}$

Why this formula? One motivation: if we have points A, B



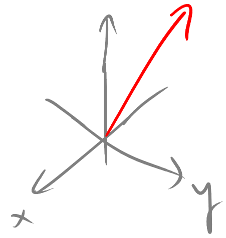
$$\vec{AB} = \langle x_2-x_1,\ y_2-y_1,\ z_2-z_1 \rangle$$

$$\|\vec{AB}\| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

$= $ distance from A to B

<u>Unit vectors</u> If $\vec{v}$ is a vector with $\|\vec{v}\|=1$, call $\vec{v}$ a <u>unit vector</u>.

<u>Ex</u> If $\vec{a} = \langle -2, 1, 3 \rangle$



what is the unit vector pointing in the same direction as $\vec{a}$?

It will be $\lambda\vec{a}$ for some λ. $\lambda\vec{a} = \langle \lambda a_1, \lambda a_2, \lambda a_3 \rangle$

$$\|\lambda\vec{a}\| = \sqrt{\lambda^2 a_1^2+\lambda^2 a_2^2+\lambda^2 a_3^2} = |\lambda|\sqrt{a_1^2+a_2^2+a_3^2} = |\lambda|\ \|a\|$$

We want $\|\lambda\vec{a}\| = 1$. So we should pick $|\lambda| = \dfrac{1}{\|a\|}$

$\|\lambda\vec{a}\| = |\lambda|\,\|a\|$

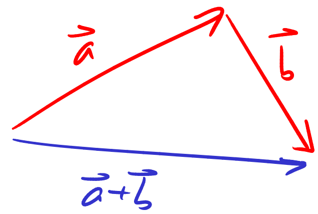
$\lambda\vec{a}$ same direction as $\vec{a}$: need $\lambda > 0$.

So take $\lambda = \dfrac{1}{\|a\|}$

In our case, $\|a\| = \sqrt{(-2)^2+1^2+3^2} = \sqrt{14}$

so $\lambda = \dfrac{1}{\sqrt{14}}$ $\quad \lambda a = \left\langle \dfrac{-2}{\sqrt{14}}, \dfrac{1}{\sqrt{14}}, \dfrac{3}{\sqrt{14}} \right\rangle$

Addition



If $\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$

$\vec{a}+\vec{b} = \langle a_1+b_1,\ a_2+b_2,\ a_3+b_3 \rangle$

Zero vector

$\vec{0} = \langle 0, 0, 0 \rangle$ in 3d

$= \langle 0, 0 \rangle$ in 2d

Basic properties

$\vec{a}+\vec{b} = \vec{b}+\vec{a}$ $\qquad (\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$

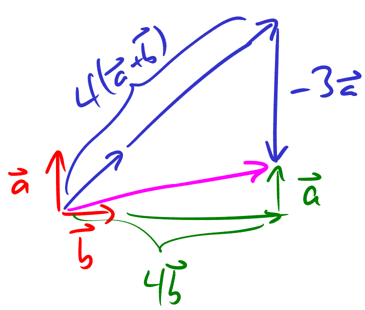
$\vec{a}+\vec{0} = \vec{a}$ $\qquad \vec{a}+(-\vec{a}) = \vec{0}$

$c(\vec{a}+\vec{b}) = c\vec{a}+c\vec{b}$ $\qquad (c+d)\vec{a} = c\vec{a}+d\vec{a}$

$(cd)\vec{a} = c(d\vec{a})$ $\qquad 1\cdot\vec{a} = \vec{a}$

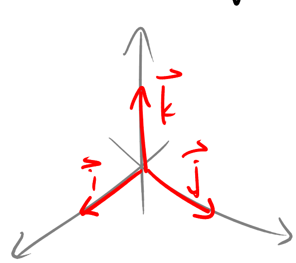
Ex

$$4(\vec{a}+\vec{b}) - 3\vec{a} = 4\vec{a}+4\vec{b}-3\vec{a} = \vec{a}+4\vec{b}$$



Basis vectors

$\vec{i} = \langle 1, 0, 0 \rangle \qquad \vec{j} = \langle 0, 1, 0 \rangle \qquad \vec{k} = \langle 0, 0, 1 \rangle$



Any vector can be built as a sum of scaler multiples of these!

Ex If $\vec{a} = \langle 1,4,-2 \rangle$ then $\vec{a} = \langle 1,0,0\rangle + \langle 0,4,0\rangle + \langle 0,0,-2\rangle$

$$= \vec{i} + 4\vec{j} - 2\vec{k}$$

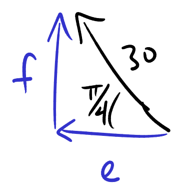
Addition of velocities

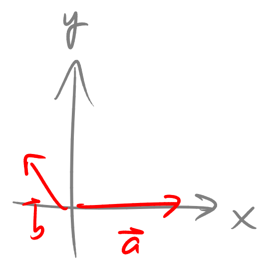
Ex A car is traveling east at 200 mph.
A fly in the car is moving northwest at 30 mph relative to the interior of the car.

What is the speed and direction of the fly as measured from outside?

It's obtained by adding the two velocity vectors:

$\vec{a} = \langle 200, 0 \rangle$





$$e = 30\cos\left(\tfrac{\pi}{4}\right) = \frac{30\sqrt{2}}{2}$$

$$f = 30\sin\left(\tfrac{\pi}{4}\right) = \frac{30\sqrt{2}}{2}$$

$$\vec{b} = \left\langle -\frac{30\sqrt{2}}{2}, \frac{30\sqrt{2}}{2} \right\rangle$$

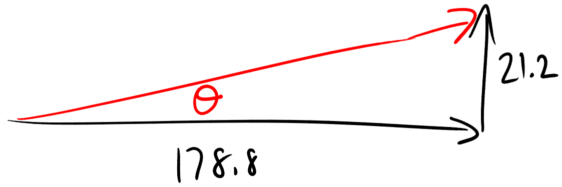
$$\approx \langle -21.2, 21.2 \rangle$$

$$\vec{c} = \vec{a} + \vec{b} = \langle 200 - 21.2,\ 0 + 21.2 \rangle$$

$$= \langle 178.8, 21.2 \rangle$$

Total speed: $\|\vec{c}\| = \sqrt{(178.8)^2 + (21.2)^2} \approx 180$ mph

Net direction:



$$\tan\theta = \frac{21.2}{178.8}$$

$$\theta = \tan^{-1}\left(\frac{21.2}{178.8}\right)$$

Dot products

$$\vec{a} = \langle a_1, a_2, a_3 \rangle \qquad \vec{b} = \langle b_1, b_2, b_3 \rangle$$

Define $\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

Ex $\langle 4, 1, \tfrac{1}{4} \rangle \cdot \langle 6, -3, -8 \rangle = 4\times 6 + 1\times(-3) + \tfrac{1}{4}\times(-8)$

$$= 24 - 3 - 2$$

$$= 19$$

(In 2 dim, $\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$

Ex $\langle 0, 4 \rangle \cdot \langle 1, 3 \rangle = 12$)

Ex $(\vec{i} \underbrace{+3\vec{k}}_{+0\vec{j}}) \cdot (\vec{i} - 2\vec{k} + \vec{j})$

$$= \langle 1, 0, 3 \rangle \cdot \langle 1, 1, -2 \rangle = 1\times 1 + 0\cdot 1 + 3\times(-2) = -5$$

Algebraic properties

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$$

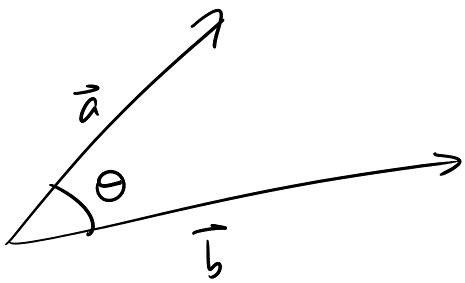
$$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$$

$$\vec{0}\cdot\vec{a} = 0$$

$$(\lambda\vec{a})\cdot\vec{b} = \lambda\cdot(\vec{a}\cdot\vec{b})$$

Ex $\quad \vec{a}\cdot\vec{a} = \langle a_1, a_2, a_3\rangle \cdot \langle a_1, a_2, a_3\rangle$

$$= a_1 a_1 + a_2 a_2 + a_3 a_3 = \|\vec{a}\|^2$$

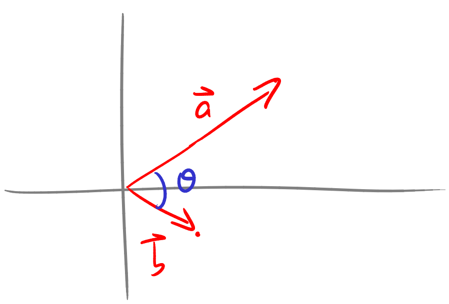


$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\cdot\|\vec{b}\|\cdot\cos\theta$$

In the plane containing both $\vec{a}$ and $\vec{b}$

Ex $\quad \vec{a} = \langle 4,3\rangle \quad \vec{b} = \langle 2,-1\rangle$

What is angle between $\vec{a}$ and $\vec{b}$?



$$\vec{a}\cdot\vec{b} = 4\times 2 + 3\cdot(-1) = 5$$

$$\|\vec{a}\| = \sqrt{4^2+3^2} = 5$$

$$\|\vec{b}\| = \sqrt{2^2+(-1)^2} = \sqrt{5}$$

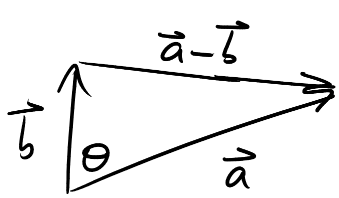
$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$$

$$5 = 5\cdot\sqrt{5}\cos\theta$$

$$\cos\theta = \frac{1}{\sqrt{5}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 1.107 \text{ radians}$$

Why is $\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$?



$\|\vec{a}\| = A$

$\|\vec{b}\| = B$

$\|\vec{a}-\vec{b}\| = C$

Law of Cosines:

$$C^2 = A^2 + B^2 - 2AB\cos\theta$$

But also, $C^2 = \|\vec{a}-\vec{b}\|^2$

$$= (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$$

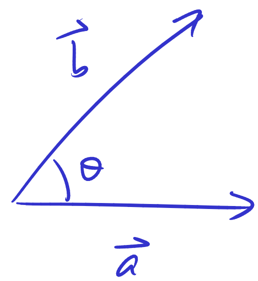
$$= \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - 2\,\vec{a}\cdot\vec{b}$$

$$= \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\,\vec{a}\cdot\vec{b}$$

$$C^2 = A^2 + B^2 - 2\,\vec{a}\cdot\vec{b}$$

So $2AB\cos\theta = 2\,\vec{a}\cdot\vec{b}$

$$\|\vec{a}\|\|\vec{b}\|\cos\theta = \vec{a}\cdot\vec{b}$$

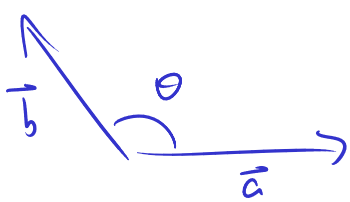


acute: $\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta > 0$

ie $\vec{a}\cdot\vec{b} > 0$

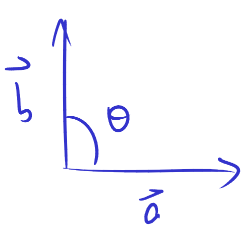
Ex $\vec{a} = \langle 1,0\rangle$ $\quad \vec{b} = \langle 3,2\rangle$

$\vec{a}\cdot\vec{b} = 3 > 0$



obtuse: $\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta < 0$

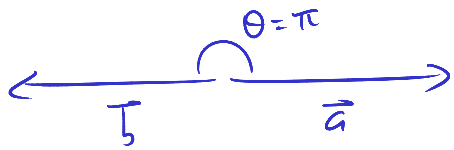
$\vec{a}\cdot\vec{b} < 0$



right: $\vec{a}\cdot\vec{b} = 0$

Ex $\vec{a} = \langle 2,1,12\rangle$ $\quad \vec{b} = \langle -2,4,0\rangle$ $\quad \vec{a}\cdot\vec{b} = 0$

so these two are <u>orthogonal</u> (make a right angle)



$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta = -\|\vec{a}\| \|\vec{b}\|$$