Lecture 6

16 Sep 2014

This course now has a Facebook group: 408M Neitzke

Last time: dot product of vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Ex

$$\vec{i} \cdot \vec{i} = \langle 1,0,0 \rangle \cdot \langle 1,0,0 \rangle = 1$$
$$\vec{j} \cdot \vec{j} = 1$$
$$\vec{k} \cdot \vec{k} = 1$$

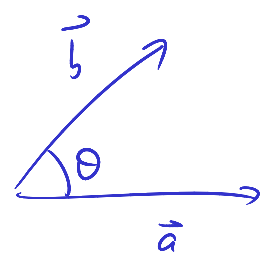
(consistent with $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$)

$$\vec{i} \cdot \vec{j} = \langle 1,0,0 \rangle \cdot \langle 0,1,0 \rangle = 0$$
$$\vec{j} \cdot \vec{k} = 0$$
$$\vec{i} \cdot \vec{k} = 0$$

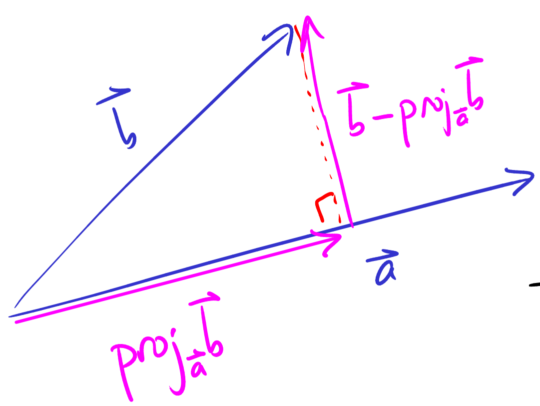
(consistent with $\vec{a} \cdot \vec{b} = 0 \iff \vec{a} \perp \vec{b}$)

$\vec{a}, \vec{b}$ perpendicular orthogonal (right angle)

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\theta$$



Projections

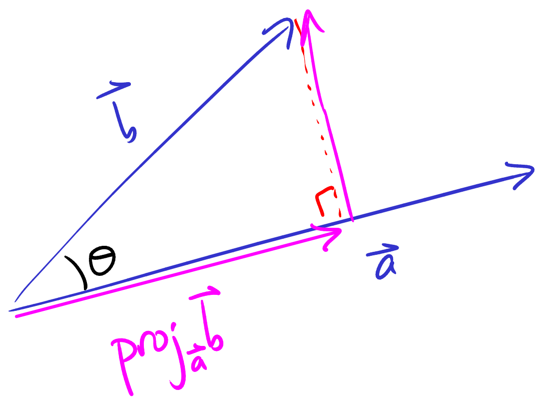


Decompose $\vec{b}$ as the sum of two pieces:

$$\vec{b} = \underbrace{\text{proj}_{\vec{a}} \vec{b}}_{\text{parallel to } \vec{a}} + \underbrace{(\vec{b} - \text{proj}_{\vec{a}} \vec{b})}_{\perp \text{ to } \vec{a}}$$

How to actually calculate $\text{proj}_{\vec{a}} \vec{b}$?

Start with θ acute:



Let $\text{comp}_{\vec{a}}\vec{b}$ mean the length $\|\text{proj}_{\vec{a}}\vec{b}\|$.

$$\cos\theta = \frac{\text{comp}_{\vec{a}}\vec{b}}{\|\vec{b}\|}$$

ie $\text{comp}_{\vec{a}}\vec{b} = \|\vec{b}\|\cos\theta$

Recall $\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$

so $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}$

$$\text{comp}_{\vec{a}}\vec{b} = \|\vec{b}\|\cdot\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\cdot\|\vec{b}\|}$$

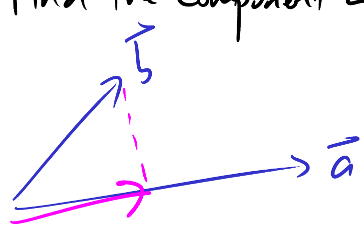
$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|}$$

Then $\text{proj}_{\vec{a}}\vec{b}$ is a vector in the direction of $\vec{a}$ with length $\text{comp}_{\vec{a}}\vec{b}$: that's

$$\text{proj}_{\vec{a}}\vec{b} = \frac{(\text{comp}_{\vec{a}}\vec{b})}{\|\vec{a}\|}\,\vec{a}$$

$$= \frac{(\vec{a}\cdot\vec{b})}{\|\vec{a}\|^2}\,\vec{a}$$

Ex Find the component and the projection of $\vec{b} = \langle 1,3,2\rangle$ along $\vec{a} = \langle 3,4,0\rangle$.

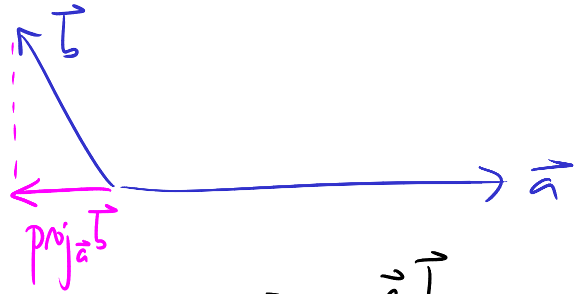


$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|} = \frac{15}{5} = 3$$

$$\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\,\vec{a} = \frac{15}{25}\langle 3,4,0\rangle$$

$$= \frac{3}{5}\langle 3,4,0\rangle = \left\langle \frac{9}{5}, \frac{12}{5}, 0\right\rangle$$

If θ not acute

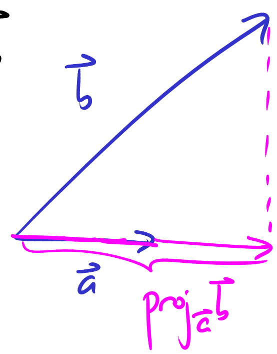


still use the same formula: $\text{proj}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\vec{a}$

$\text{comp}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|}$ will be negative in the case θ obtuse

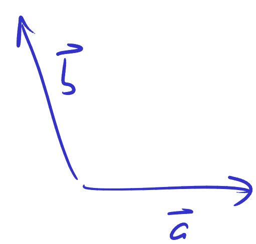
(then it's minus the length of $\text{proj}_{\vec{a}}\vec{b}$)

[It's OK if $\text{proj}_{\vec{a}}\vec{b}$ is longer than $\vec{a}$!]



Ex Find the component of $\vec{b} = \langle -2, 4\rangle$ along $\vec{a} = \langle 3, 0\rangle$.

$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|} = \frac{-6}{3} = \underline{\underline{-2}}$$



The Cross Product (Ch 12.4)

Another way of "multiplying" vectors: this time the product of 2 vectors will be another vector! Only works in 3-d

$\vec{a} = \langle a_1, a_2, a_3\rangle \quad \vec{b} = \langle b_1, b_2, b_3\rangle \qquad$ new vector $\vec{c} = \vec{a}\times\vec{b}$

$$\vec{a}\times\vec{b} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\rangle$$

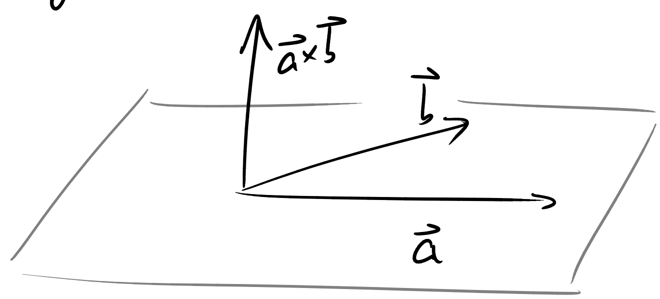
Geometric interpretation:

① $\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\,\|\vec{b}\|\sin\theta$

(cf. $\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$)

② $\vec{a}\times\vec{b}$ is $\perp$ to both $\vec{a}$ <u>and</u> $\vec{b}$

with orientation determined by right hand rule:



<u>Rk</u> This means $\vec{a}\times\vec{b}$ is <u>not</u> the same as $\vec{b}\times\vec{a}$!

In fact, $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

<u>Determinants</u> For a 2×2 matrix:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$$

$-$ $+$

<u>Ex</u> $\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = 3\times6-4\times5 = 18-20 = -2$

For a 3×3 matrix:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$+ a_3 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Ex

$$\begin{vmatrix} 1 & 3 & 5 \\ -2 & -1 & 4 \\ 3 & 1 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} -1 & 4 \\ 1 & 1 \end{vmatrix} - 3 \begin{vmatrix} -2 & 4 \\ 3 & 1 \end{vmatrix} + 5 \begin{vmatrix} -2 & -1 \\ 3 & 1 \end{vmatrix}$$

$$= 1 \cdot (-1-4) - 3 \cdot (-2-12) + 5(-2+3)$$

$$= -5 + 42 + 5 = \underline{\underline{42}}$$

Cross product

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

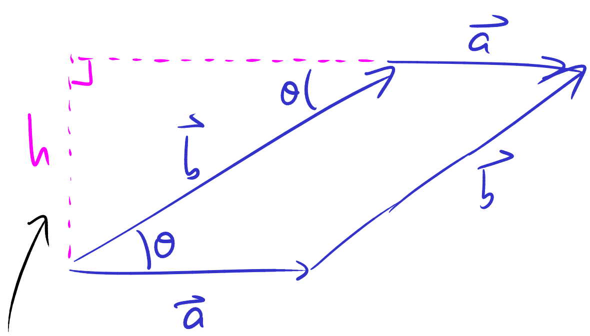
Ex $\vec{a} = \langle -1, 2, 2 \rangle \quad \vec{b} = \langle 3, 0, -1 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 2 \\ 3 & 0 & -1 \end{vmatrix} = \vec{i} \begin{vmatrix} 2 & 2 \\ 0 & -1 \end{vmatrix} - \vec{j} \begin{vmatrix} -1 & 2 \\ 3 & -1 \end{vmatrix} + \vec{k} \begin{vmatrix} -1 & 2 \\ 3 & 0 \end{vmatrix}$$

$$= \vec{i}(-2) - \vec{j}(-5) + \vec{k}(-6)$$
$$= -2\vec{i} + 5\vec{j} - 6\vec{k}$$
$$= \langle -2, 5, -6 \rangle$$

<u>Check</u>: $(\vec{a} \times \vec{b}) \cdot \vec{a} = \langle -2, 5, -6 \rangle \cdot \langle -1, 2, 2 \rangle$
$$= 2 + 10 - 12 = \underline{0}$$ ✓
$$(\vec{a} \times \vec{b}) \cdot \vec{b} = \langle -2, 5, -6 \rangle \cdot \langle 3, 0, -1 \rangle$$
$$= -6 + 0 + 6 = \underline{0}$$

Areas of parallelograms

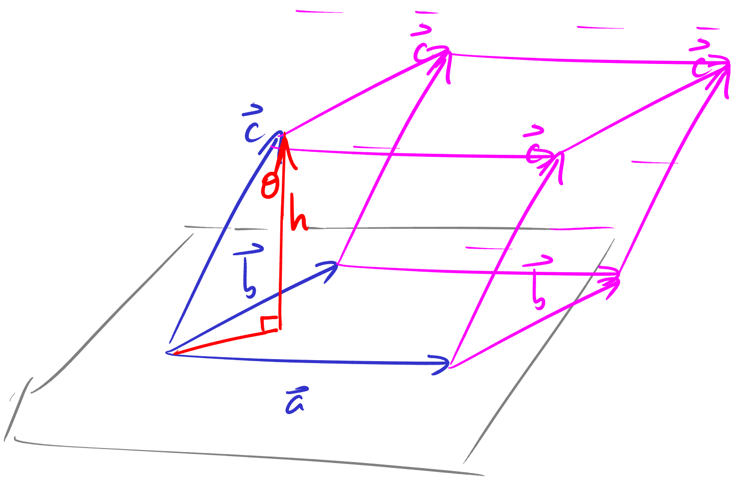


$$\text{Area} = \text{base} \cdot \text{height}$$
$$= \|\vec{a}\| \|\vec{b}\| \sin\theta$$
$$= \|\vec{a} \times \vec{b}\|$$

$\sin\theta = \dfrac{h}{\|\vec{b}\|}$ so $h = \|\vec{b}\| \sin\theta$

(So, in particular, if $\vec{a}$ and $\vec{b}$ are parallel then this area is zero, consistent with the fact that if $\vec{a}$ and $\vec{b}$ are parallel then $\vec{a} \times \vec{b} = 0$)

Volumes of parallelepipeds



Given $\vec{a}, \vec{b}, \vec{c}$ we make a "crystal" (parallelepiped) in 3 dimensions.

What is its volume?

Volume = (area of base) × (height)

distance from top face to bottom face

$$h = \|\vec{c}\| \cdot \cos\theta$$

But θ is the angle between $\vec{c}$ and $\vec{a}\times\vec{b}$!

$$V = \|\vec{a}\times\vec{b}\|\,\|\vec{c}\|\cos\theta$$

So, $$V = \left|(\vec{a}\times\vec{b})\cdot\vec{c}\right|$$

$$= \left| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \cdot \langle c_1, c_2, c_3 \rangle \right|$$

$$= \left| \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \right|$$

Ex Show that the vectors $\vec{a} = \langle 1, 4, -7 \rangle$, $\vec{b} = \langle 2, -1, 4 \rangle$, $\vec{c} = \langle 0, -9, 18 \rangle$ are coplanar.

$$V = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = 1 \cdot \begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4 \cdot \begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} + (-7) \begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix}$$

$$= 1 \cdot 18 - 4 \cdot 36 + (-7) \cdot (-18)$$

$$= 8 \cdot 18 - 4 \cdot 36 = \underline{\underline{0}}$$