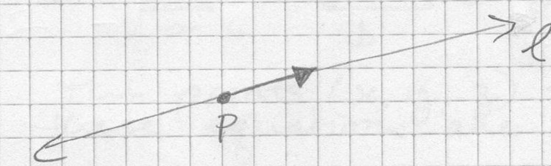
Lecture 7 — 18 September 2014

Equations of Lines and Planes

Line in x-y plane: defined by a point & slope.

In 3-d space, we have a point and a direction vector:

A line ℓ is given by a point on the line and a direction vector indicating direction along the line.



Remarks:

1) If P, Q are pts on the line, then $\vec{PQ}$ is a direction vector

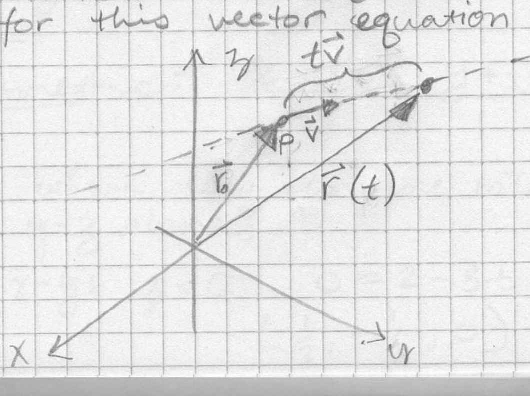
2) a scalar multiple of a direction vector is also a direction vector.

3) Two lines are parallel if a direction vector for one is a scalar multiple of a direction vector of another.

Given $P(x_0, y_0, z_0)$ and direction vector $\vec{v} = \langle a, b, c \rangle$, the corresponding line ℓ is described by the vector equation:

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$

Where $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ and t is a real number $-\infty < t < \infty$. Call t the parameter for this vector equation



$P(x_0, y_0, z_0)$

$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$

$$\vec{r}(t) = \vec{r}_0 + t\vec{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

continued...

$L = \{ (x, y, z) \mid x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct,\ -\infty < t < \infty \}$

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$ Parametric equations

Alternatively: Write x, y, z in terms of t

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

The points (x, y, z) that satisfy these equations are on the line.

These are symmetric equations for ℓ.

Ex: Find the vector, parametric and symmetric equations for line ℓ with direction vector $\vec{v} = \langle -2, 1, -3 \rangle$ and through the point $P(1, -1, 2)$

vector:
$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$
$$= \langle 1, -1, 2 \rangle + t\langle -2, 1, -3 \rangle$$
$$= \langle 1 - 2t, -1 + t, 2 - 3t \rangle$$

parametric:
$$\begin{cases} x = 1 - 2t \\ y = -1 + t \\ z = 2 - 3t \end{cases}$$

symmetric: $$\frac{x - 1}{-2} = \frac{y + 1}{1} = \frac{z - 2}{-3}$$

Q: where does this line intersect the x-y plane? y-z plane?

x-y: $z = 0$ $\quad 0 = 2 - 3t \rightarrow t = \frac{2}{3}$ $\quad x = 1 - \frac{4}{3} = -\frac{1}{3}$

$\left(-\frac{1}{3}, -\frac{1}{3}, 0\right)$ $\quad y = -1 + \frac{2}{3} - 1/3$

continued...

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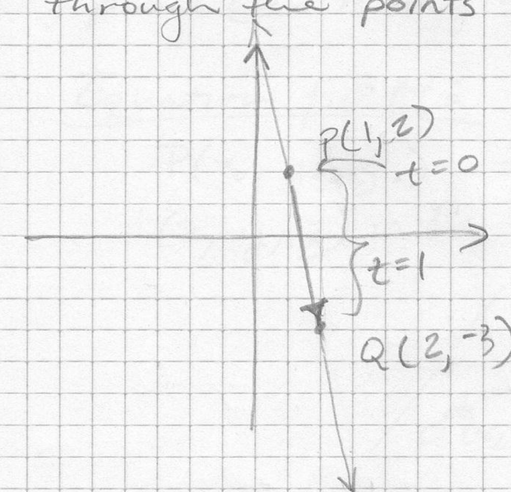
y-z: $x = 0 \rightarrow t = \frac{1}{2}$

$y = -\frac{1}{2}$ $\quad (0, -\frac{1}{2}, \frac{1}{2})$

$z = \frac{1}{2}$

$\underline{\mathbb{R}^2}$: All this works to describe lines in the plane

Example: find the marametric equation for line through the points $(1,2)$, $(2,-3)$



$\vec{v} = \overrightarrow{PQ} = \langle 1, -5 \rangle$

$\vec{r} = \vec{r}_0 + t\vec{v}$

$= \langle 1, 2 \rangle + t \langle 1, -5 \rangle$

$= \langle 1+t, 2-5t \rangle$

$$\begin{cases} x = 1+t \\ y = 2-5t \end{cases}$$

If we had used Q instead of P, we would get the same line, but a different parameterizatio

For what values of t does the parametric equation describe the line segment $\overline{PQ}$ between P and Q?

$0 \leq t \leq 1$

Parameterize the points on the line segment:

$x = 1+t$
$y = 2-5t$ $\quad 0 \leq t \leq 1$

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Equations of Planes in $\mathbb{R}^3$

Principle: a plane $\mathcal{P}$ is determined by a point on the plane and a direction perpendicular (orthogonal) to the plane.

Let $\vec{n} = \langle a, b, c\rangle$ $P(x_0, y_0, z_0)$

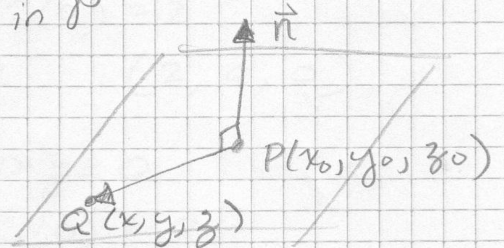
There is an unique plane $\mathcal{P}$ through P_0 that is perpendicular to $\vec{n}$.

$\vec{n}$ is called a normal vector to the plane (as is any scalar multiple of $\vec{n}$)

Equation for $\mathcal{P}$:

$P(x_0, y_0, z_0)$ $\vec{n} = \langle a, b, c\rangle$

(x, y, z) in $\mathcal{P}$



$$\vec{PQ} \cdot \vec{n} = 0$$

$$\langle x - x_0, y - y_0, z - z_0\rangle \cdot \langle a, b, c\rangle = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Thm: the equation the plane through the point (x_0, y_0, z_0) with $\vec{n} = \langle a, b, c\rangle$ is:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz = k \quad \text{where } k = ax_0 + by_0 + cz_0$$

Note: You can read off a normal vector for plane from coefficients of eqn.

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Ex: Find the plane $\mathcal{P}$ through $P_0(1,1,1)$ with normal vector $\vec{n}\langle 1,-1,2\rangle$.
Where does $\mathcal{P}$ intersect the x-axis? y-axis? z-axis? Sketch the plane.

Equation: $\langle x-1, y-1, z-1\rangle \cdot \langle 1,-1,2\rangle = 0$

$$1(x-1)+-1(y-1)+2(z-1)=0$$
$$x-y+2z=2$$

x-axis: $y=0 \quad z=0$

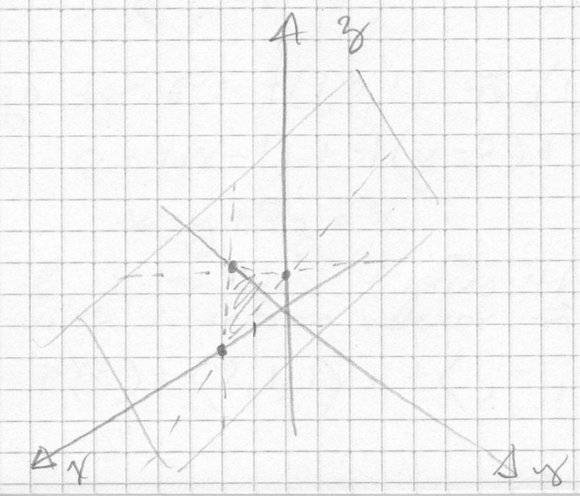
$$x-0+2(0)=2$$
$$x=2 \qquad (2,0,0)$$

y-axis: $x=0 \quad z=0$

$$0-y+2(0)=2$$
$$y=-2 \qquad (0,-2,0)$$

z-axis: $x=0, \quad y=0$

$$0-0+2z=2 \qquad (0,0,1)$$
$$z=1$$



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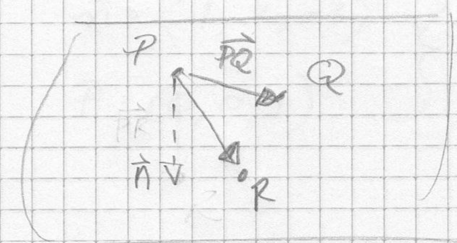
<u>Ex</u>: Let $\mathcal{P}$ be the plane containing

$P(1,0,1)$, $Q(2,1,1)$, $R(-1,5,2)$

a) Find eqn of $\mathcal{P}$

b) Find eqn of line perpendicular to $\mathcal{P}$ through $Q(2,1,1)$



Cross Product gives vector perpendicular to plane

$$\overrightarrow{PQ} = \langle 1, 1, 0 \rangle$$
$$\overrightarrow{PR} = \langle -2, 5, 1 \rangle$$

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ -2 & 5 & 1 \end{vmatrix} = \vec{i} \begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 1 \\ -2 & 5 \end{vmatrix}$$

$$= \vec{i}(1-0) - \vec{j}(1-0) + \vec{k}(5+2)$$
$$= \langle 1, -1, 7 \rangle$$

eqn for $\mathcal{P}$: $P(1,0,1)$ $\vec{n}(1,-1,7)$

$$0 = 1(x-1) + -1(y-0) + 7(z-1)$$
$$x - y + 7z = 8$$

b) $\ell \perp$ to Q: ℓ goes through $Q(2,1,1)$ with direction vector $\vec{n} = \langle 1, -1, 7 \rangle$

$$\vec{r}(t) = \langle 2, 1, 1 \rangle + t \langle 1, -1, 7 \rangle$$

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Definition: Two planes are parallel if their direction vectors are parallel.

Ex: Find the equation of the plane parallel to $x+2y-3z=6$ through $P(0,0,1)$

$$\left.\begin{array}{l}\vec{n}=\langle 1,2,-3\rangle \\ P=(0,0,1)\end{array}\right\}$$ can be found from these.