Lecture 8

23 Sep 2014

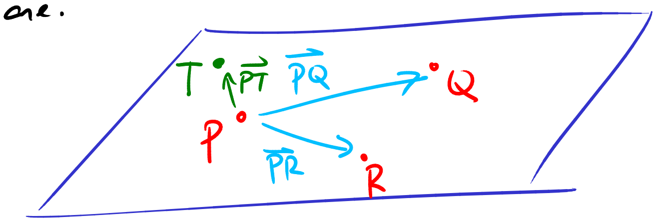
<u>Midterm 1</u>: next Thursday, in class
covers material through <u>Lecture 9</u>

Notes from Lecture 7 now posted

Last time: equations of <u>lines</u> and <u>planes</u> in 3 dimensions

<u>Ex</u>: Given 3 generic points P, Q, R how to find the plane passing through P, Q and R?

Key trick: want to find <u>normal vector</u> to the plane.
Do this by taking, say,



$$\vec{n} = \vec{PQ} \times \vec{PR}$$

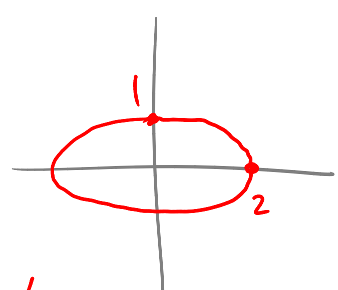
Then, the plane is the set of all points T such that $\vec{PT} \perp \vec{n}$

ie $$\vec{PT} \cdot \vec{n} = 0$$

Cylinders + Quadric Surfaces (Ch 12.6)

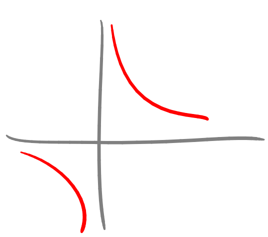
Recall, in 2 dimensions if we have $f(x,y) = Ax^2 + By^2 + Cxy + Dx + Ey + F$
the locus defined by the equation $f(x,y) = 0$
is a <u>conic section</u>.

For <u>generic</u> A, B, C, D, E, F it's either: ellipse

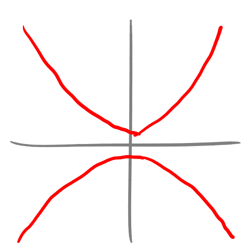


$$\frac{x^2}{4} + y^2 = 1$$

hyperbola

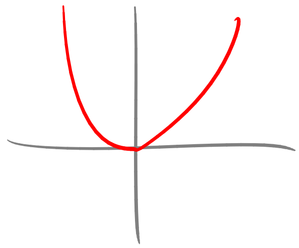


$xy = 1$

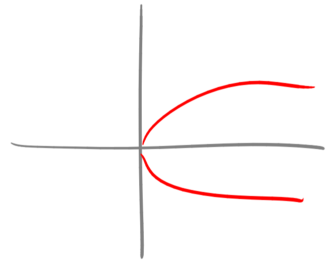


$y^2 - x^2 = 1$

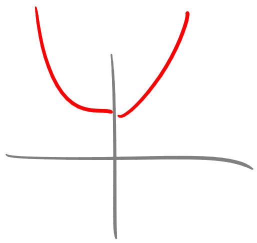
parabola



$y = x^2$



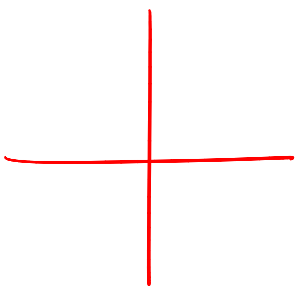
$x = y^2$



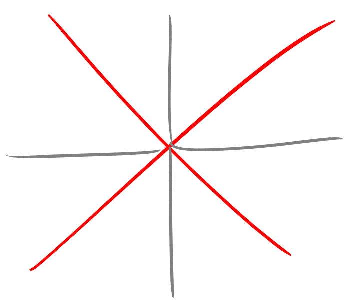
$y = 4 + x^2$

Try: $y = (x+y)^2$!

For special A, B, C, D, E, F could get something special: e.g.



$xy = 0$ 2 intersecting lines



$$x^2 - y^2 = 0$$
$$(x+y)(x-y) = 0$$

Now let's consider the analogous question in 3 dimensions.

$$f(x,y,z) = 0$$

where $f(x,y,z) = Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J$

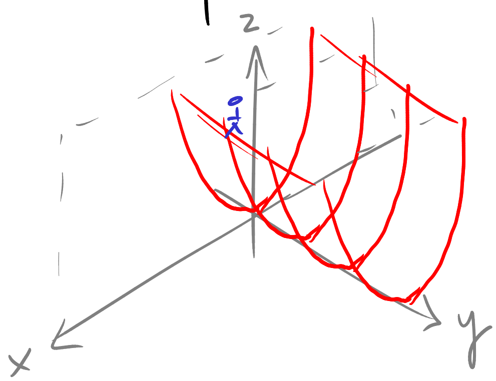
("Quadric surfaces")

Ex Sketch the surface given by $z = x^2$

Note, y doesn't appear anywhere in the equation!

Look at the planes $y = k$ for k fixed.

In each such plane we have a parabola $z = x^2$.



"halfpipe"

or parabolic cylinder

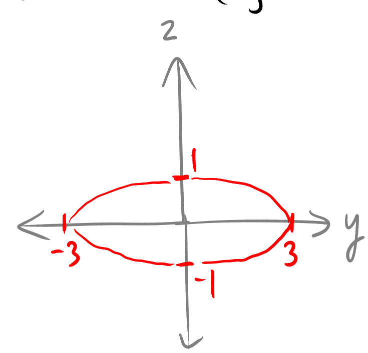
(obtained by taking a single parabola and shifting it by an arbitrary amount in the y-direction)

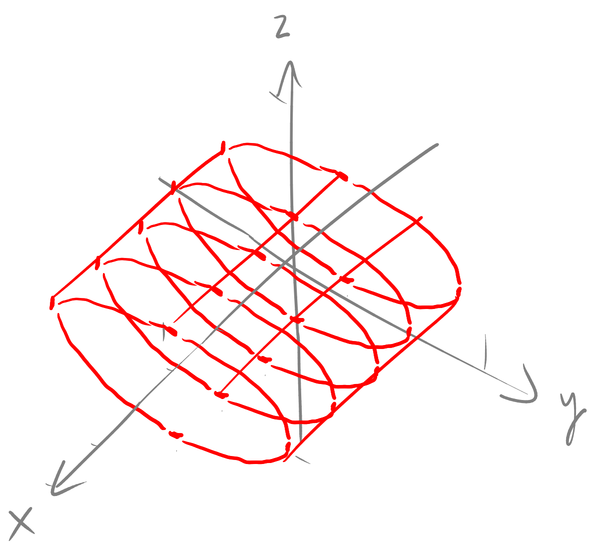
Ex Sketch the surface given by $\frac{y^2}{9} + z^2 = 1$

Note, x doesn't appear in the equation.

Look at the plane $x = k$ where k is constant (e.g. $k=0$ is yz-plane)

Then have an ellipse





elliptic cylinder

Ex Sketch the graph of $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$

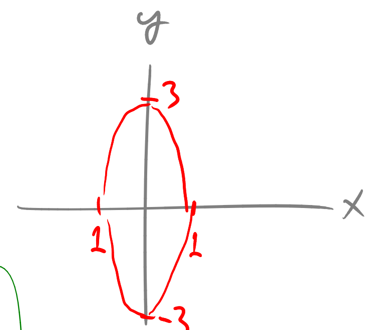
Try to understand this by looking at its intersection with various planes ("traces")

For example, say we fix $z = k$ (k constant)

Then, have $x^2 + \frac{y^2}{9} + \frac{k^2}{4} = 1$

$$x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4}$$

For example, if $k=0$, this is $x^2 + \frac{y^2}{9} = 1$



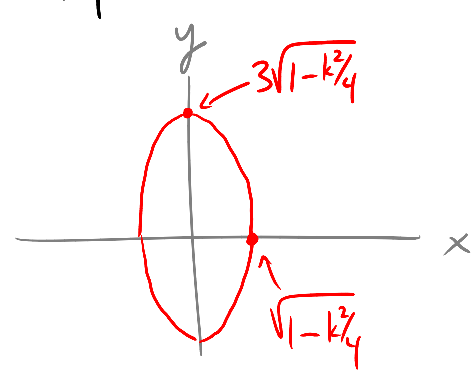
$$\frac{x^2}{1^2} + \frac{y^2}{3^2} = 1$$

ie $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $a = 1$, $b = 3$

For general k, it's $$\frac{x^2}{1 - k^2/4} + \frac{y^2}{9(1 - k^2/4)} = 1$$

if $1 - k^2/4 > 0$ this is an ellipse, with $a^2 = 1 - k^2/4$ $b^2 = 9(1 - k^2/4)$

$$a = \sqrt{1 - k^2/4} \qquad b = 3\sqrt{1 - k^2/4}$$



As k increases from 0 to 2, the ellipse shrinks

at $k=2$ it shrinks to a single point

Similarly as k decreases from 0 to -2, ellipse shrinks

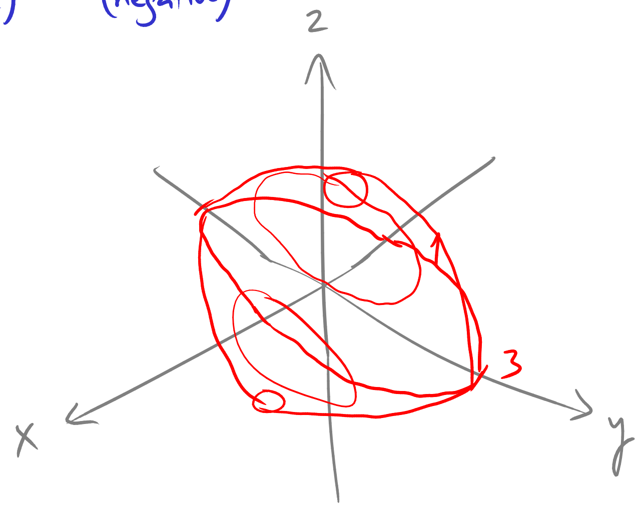
at $k=2$, shrinks to a single point

If $k>2$, our equation

$$\frac{x^2}{1-k^2/4}+\frac{y^2}{9(1-k^2/4)}=1$$

is

$$\frac{x^2}{(\text{negative})}+\frac{y^2}{(\text{negative})}=1 \implies \underline{\text{no solutions}} \text{ for } x,y!$$



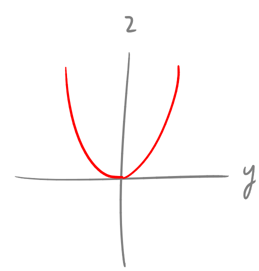
Similarly could have looked at traces $x=k$ or $y=k$

— get ellipses in each case, just like we got above.

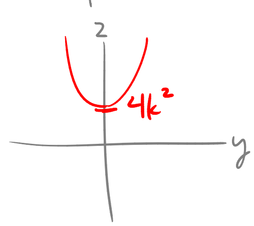
Good idea to look at traces in all 3 directions.

<u>Ex</u> Sketch $z=4x^2+y^2$

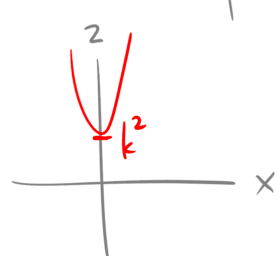
Take trace in plane $x=0$: $z=y^2$



in plane $x=k$: $z=4k^2+y^2$



in plane $y=k$: $z=4x^2+k^2$



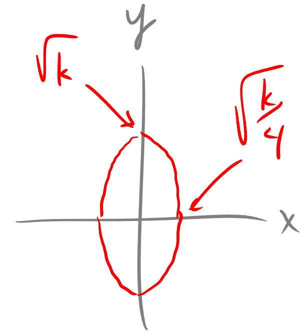
in plane $z=k$: $k=4x^2+y^2$

$$1=\frac{4x^2}{k}+\frac{y^2}{k}$$

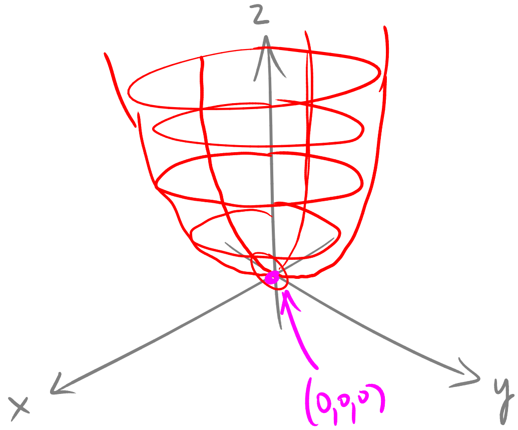
$$1=\frac{x^2}{\left(\frac{k}{4}\right)}+\frac{y^2}{k}$$

<u>no solutions</u> if $k<0$

ellipse if $k>0$: $1=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ $\quad a=\sqrt{\frac{k}{4}}\quad b=\sqrt{k}$



Put it together:



<u>elliptic paraboloid</u>

<u>Rk</u> We just saw that $z=4x^2+y^2$ gives an elliptic paraboloid with vertex $(0,0,0)$.

Similarly $y=4x^2+z^2$ " " " " " " $(0,0,0)$

Similarly $y-3=4(x+7)^2+(z-1)^2$ is an elliptic paraboloid with vertex $(-7,3,1)$.

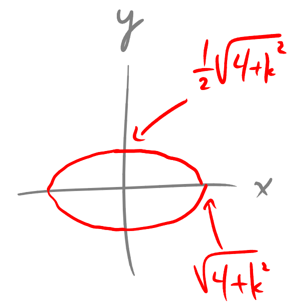
<u>Ex</u> Sketch the locus given by $\frac{x^2}{4}+y^2-\frac{z^2}{4}=1$.

Trace in plane $z=k$: $\frac{x^2}{4}+y^2-\frac{k^2}{4}=1$

$$\frac{x^2}{4}+y^2=1+\frac{k^2}{4}$$

$$\frac{x^2}{4+k^2} + \frac{y^2}{1+\frac{k^2}{4}} = 1$$

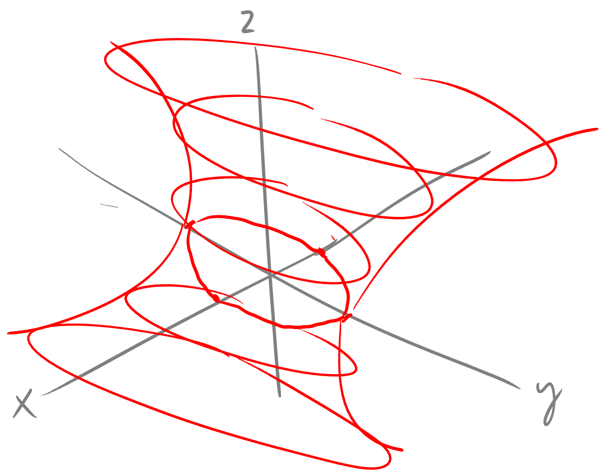
ellipse with $a = \sqrt{4+k^2}$ $\quad b = \sqrt{1+\frac{k^2}{4}} = \frac{1}{2}\sqrt{4+k^2}$



NB, this ellipse grows as k does!

Trace in plane $x=0$: $\quad y^2 - \frac{z^2}{4} = 1 \quad$ hyperbola

in plane $y=0$: $\quad \frac{x^2}{4} - \frac{z^2}{4} = 1 \quad$ hyperbola



hyperboloid of one sheet