Lecture 9

25 Sep 2014

Midterm 1 next Thursday (1 week from today) in-class
including material thru Lecture 9

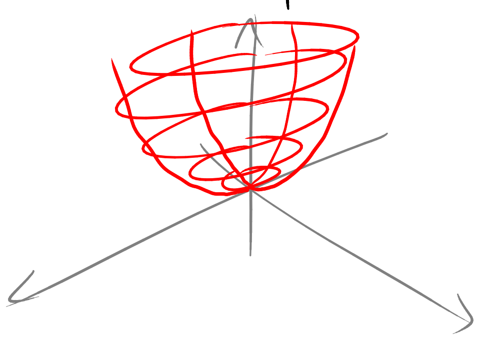
Last time: cylinders + quadric surfaces in 3 dimensions

Ex $z = x^2$ parabolic cylinder (all traces in planes $y=k$ are parabolas $z=x^2$)

$y^2+z^2=1$ cylinder (all traces in planes $x=k$ are circles $y^2+z^2=1$)

$x^2+\frac{y^2}{9}+\frac{z^2}{4}=1$ ellipsoid (traces in planes $x=k$, $y=k$, $z=k$ are all ellipses or nothing)

$z = 4x^2+y^2$ elliptic paraboloid (traces $x=k$ parabola, $y=k$ parabola, $z=k$ ellipse if $k>0$, nothing if $k<0$)



$\frac{x^2}{4}+y^2-\frac{z^2}{4}=1$

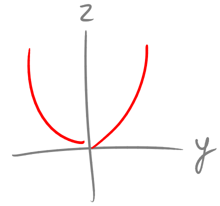
(traces $z=k$ ellipse, $x=0$ hyperbola, $y=0$ hyperbola)

hyperboloid 1 sheet

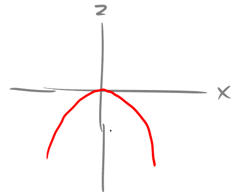
Quadric Surfaces Cont'd

Sketch the locus $z = y^2 - x^2$.

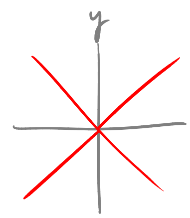
Traces: in plane $x=0$, get $z = y^2$



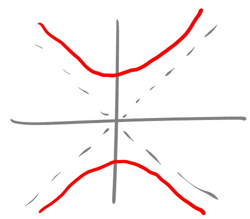
in plane $y=0$, get $z = -x^2$



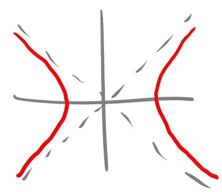
in plane $z=0$, get $0 = y^2 - x^2$
$= (y-x)(y+x)$



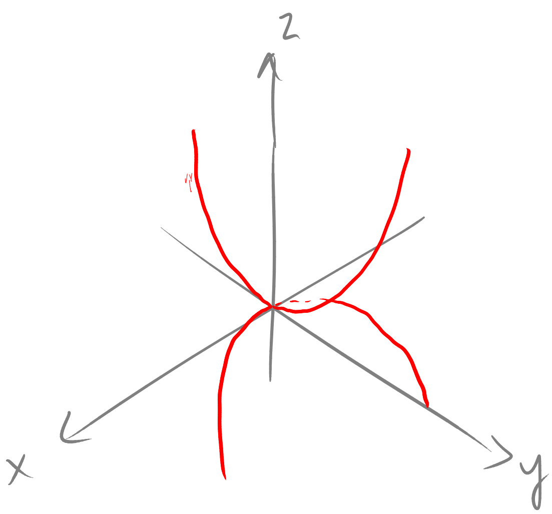
in plane $z=k$, get $k = y^2 - x^2$



$k>0$



$k<0$



hyperbolic paraboloid

Ex Sketch the locus $4x^2 - y^2 + 2z^2 + 4 = 0$

Traces: $x=0$ — $-y^2 + 2z^2 + 4 = 0$ hyperbola
$z=0$ — $4x^2 - y^2 + 4 = 0$ hyperbola
$y=0$ — $4x^2 + 2z^2 + 4 = 0$

$$4x^2 + 2z^2 = -4 \qquad \text{nothing!}$$

$y = k$ — $$4x^2 - k^2 + 2z^2 + 4 = 0$$

$$4x^2 + 2z^2 = k^2 - 4$$

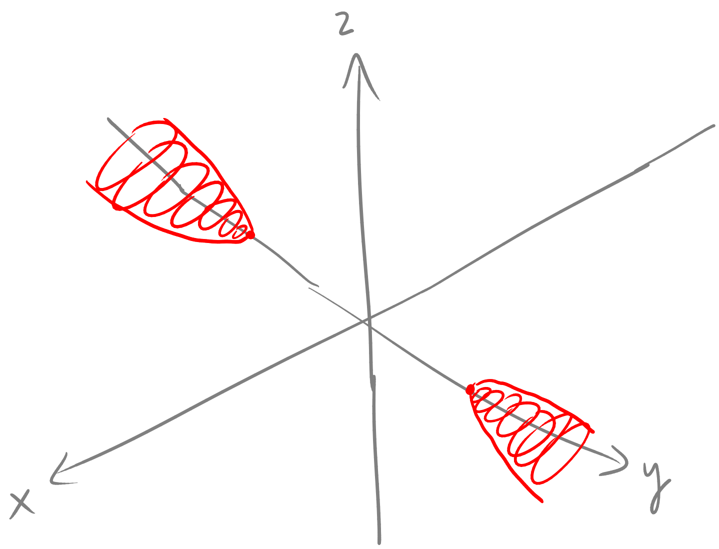
$$\frac{4x^2}{k^2-4} + \frac{2z^2}{k^2-4} = 1$$

$$\frac{x^2}{\left(\frac{k^2}{4}-1\right)} + \frac{z^2}{2\left(\frac{k^2}{4}-1\right)} = 1$$

if $k^2/4 - 1 > 0$, i.e. if $k^2 > 4$, this is an ellipse $\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$

$$a = \sqrt{\tfrac{k^2}{4} - 1} \qquad b = \sqrt{2\left(\tfrac{k^2}{4} - 1\right)}$$

if $k^2/4 - 1 < 0$, ie if $k^2 < 4$, this is nothing (empty set)



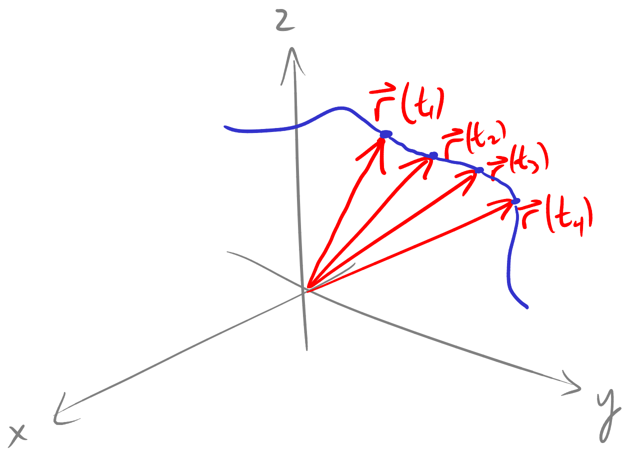
hyperboloid of 2 sheets

Summary table of quadric surfaces in text p. 830

Vector Functions (Ch 13.1)

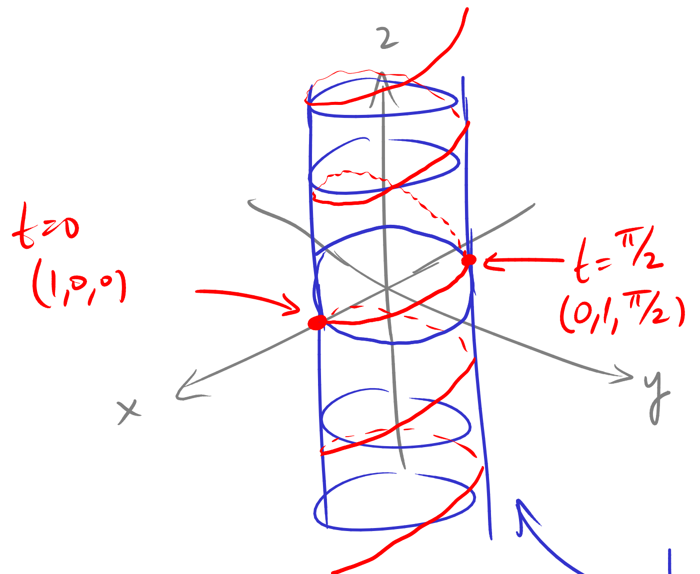
A vector function is a function of the form $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

Vector functions can be viewed as <u>parametrized curves</u> in 3 dimensions.



for every t, put the tail of the vector $\vec{r}(t)$ at $(0,0,0)$, then the tips of $\vec{r}(t)$ sweep out a curve.

<u>Ex</u> Sketch $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ as a param. curve in 3 dim.



First 2 coords $\langle \cos t, \sin t \rangle$ lie on unit circle $\Longrightarrow$ viewed from overhead, see particle going around circle as t goes from 0 to 2π.

<u>Limits</u> If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

we define $$\lim_{t \to a} \vec{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle$$

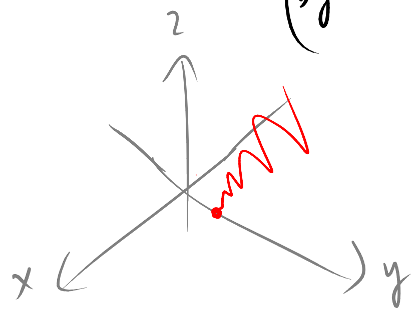
If any of $\lim_{t\to a} f(t)$, $\lim_{t\to a} g(t)$, $\lim_{t\to a} h(t)$ <u>don't</u> exist,

we say $\lim_{t\to a} \vec{r}(t)$ doesn't exist.

Ex Say $\vec{r}(t) = \langle t, \frac{\sin t}{t}, t \ln t \rangle$

$$\lim_{t \to 0} \vec{r}(t) = \left\langle \lim_{t\to 0} t, \lim_{t \to 0} \frac{\sin t}{t}, \lim_{t\to 0} t \ln t \right\rangle$$

$$= \langle 0, 1, 0 \rangle \quad \left(\text{by L'Hospital: using } t \ln t = \frac{\ln t}{\left(\frac{1}{t}\right)}\right)$$



We say that $\vec{r}(t)$ is <u>continuous</u> if, for every a (in the domain of $\vec{r}(t)$),

$$\lim_{t \to a} \vec{r}(t) = \vec{r}(a).$$

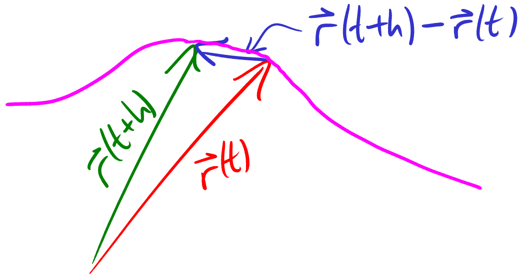
(If $\vec{r}(t)$ is continuous then the corresponding param. curve in 3 dim can be drawn without lifting the pencil)

<u>Derivatives and integrals of vector functions</u> (Ch 13.2)

By definition,

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

What does this <u>mean</u> geometrically?



As $h \to 0$, the direction of

$$\vec{r}(t+h)-\vec{r}(t)$$

approaches a tangent direction to the curve swept out by $\vec{r}(t)$.

Thus, if $\displaystyle\lim_{h\to 0} \frac{\vec{r}(t+h)-\vec{r}(t)}{h}$ exists and is not $\vec{0}$

it gives a vector tangent to the curve swept out by $\vec{r}(t)$.

This is our geometric interpretation of $\vec{r}\,'(t)$.

How to calculate $\vec{r}\,'(t)$:

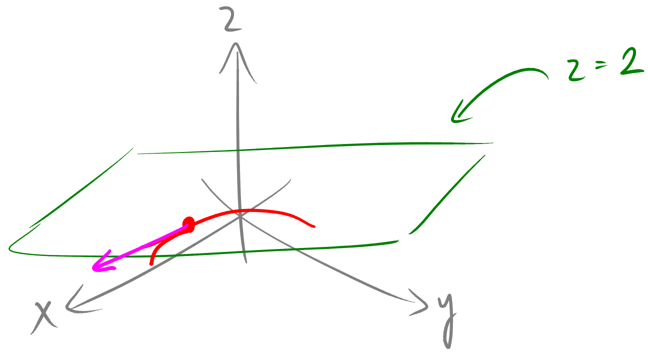
If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ then $\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t)\rangle$

Why?
$$\begin{aligned}
\vec{r}\,'(t) &= \lim_{h\to 0} \frac{1}{h}\left(\vec{r}(t+h)-\vec{r}(t)\right)\\
&= \lim_{h\to 0} \frac{1}{h}\langle f(t+h)-f(t),\ g(t+h)-g(t),\ h(t+h)-h(t)\rangle\\
&= \left\langle \lim_{h\to 0}\frac{f(t+h)-f(t)}{h},\ \lim_{h\to 0}\frac{g(t+h)-g(t)}{h},\ \lim_{h\to 0}\frac{h(t+h)-h(t)}{h}\right\rangle\\
&= \langle f'(t), g'(t), h'(t)\rangle
\end{aligned}$$

Ex Say $\vec{r}(t) = \langle 4t^2, \frac{1}{t}, 2\rangle$

Find $\vec{r}\,'(t)$ at $t=1$.

$$\vec{r}\,'(t) = \langle 8t, -\frac{1}{t^2}, 0\rangle \quad \text{so} \quad \vec{r}\,'(1) = \langle 8, -1, 0\rangle$$



The magnitude of $\vec{r}\,'(t)$ has information about "how fast" this parameterized curve is going at time t.

Sometimes we don't care about that: just want a convenient choice of tangent vector — then take unit tangent vector, a tangent vector $\vec{T}(t)$ with $\|\vec{T}(t)\| = 1$.

Obtained by:
$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

Ex If $\vec{r}(t) = \langle 4t^2, \frac{1}{t}, 2\rangle$ a unit tangent vector at $t=1$ is given by:

$$\vec{r}\,'(1) = \langle 8, -1, 0\rangle \qquad \text{(computed above)}$$

Then $\|\vec{r}\,'(1)\| = \sqrt{8^2 + (-1)^2 + 0^2} = \sqrt{65}$

So unit t.v. is $\vec{T}(t) = \dfrac{\langle 8, -1, 0\rangle}{\sqrt{65}} = \left\langle \frac{8}{\sqrt{65}}, \frac{-1}{\sqrt{65}}, 0\right\rangle$