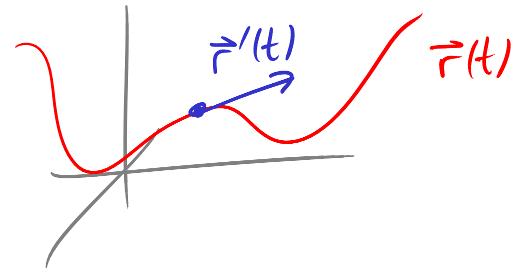
Lecture 11

17 Oct 2014

Last time: Derivatives and integrals of vector functions

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$



$\vec{r}\,'(t)$ is velocity of a particle moving with displacement $\vec{r}(t)$ from origin

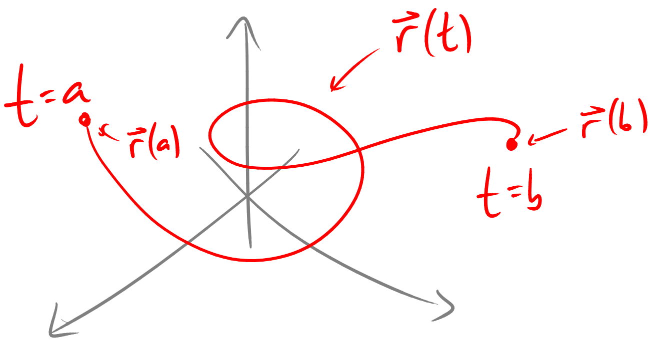
$\|\vec{r}\,'(t)\|$ is speed of the particle

If $\vec{v}(t) = \langle a(t), b(t), c(t) \rangle$

$$\int \vec{v}(t)\,dt = \left\langle \int a(t)\,dt, \int b(t)\,dt, \int c(t)\,dt \right\rangle$$

FTC: if $\dfrac{d\vec{R}}{dt} = \vec{r}(t)$ then $\displaystyle\int_a^b \vec{r}(t)\,dt = \vec{R}(b) - \vec{R}(a)$

Arc length + Curvature (Ch 13.3)



$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

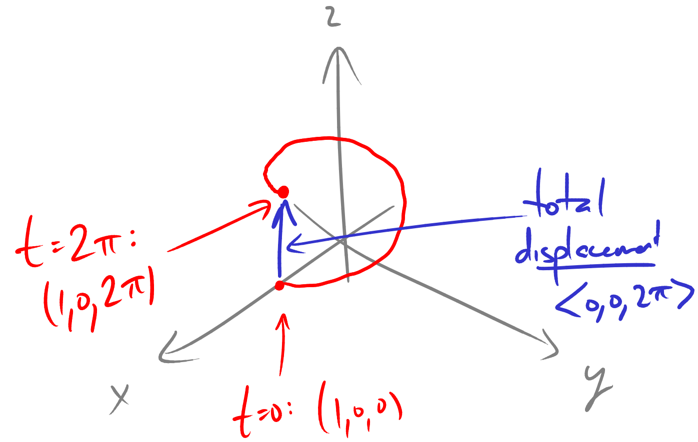
Arc length:

$$L = \int_a^b \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}\,dt$$

$$= \int_a^b \|\vec{r}\,'(t)\|\,dt$$

Ex Helix $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

Arc length from $t=0$ to $t=2\pi$:



$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1}$$
$$= \sqrt{2}$$

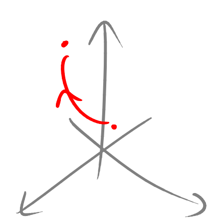
$$L = \int_0^{2\pi} \|\vec{r}\,'(t)\|\, dt = \int_0^{2\pi} \sqrt{2}\, dt = \underline{\underline{2\pi \cdot \sqrt{2}}}$$

NB:
$$\int_0^{2\pi} \vec{r}\,'(t)\, dt = \vec{r}(2\pi) - \vec{r}(0) \quad \text{(total displacement)}$$
$$= \langle 1,0,2\pi \rangle - \langle 1,0,0 \rangle$$
$$= \langle 0,0,2\pi \rangle$$

Reparameterization

Consider the curve $\vec{r}_1(t) = \langle t, t^2, t^3 \rangle$ $\quad 1 \le t \le 2$

path from $\langle 1,1,1 \rangle$ to $\langle 2,4,8 \rangle$



We could equally well describe this curve by

$$\vec{r}_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle \qquad 0 \le u \le \ln 2$$

path from $\langle 1,1,1 \rangle$ to $\langle 2,4,8 \rangle$

$\vec{r}_1(t)$ and $\vec{r}_2(u)$ are related by change of variable $t = e^u$

The arc length $L = \int_1^2 \|\vec{r}_1'(t)\|\,dt$ equals $L = \int_0^{\ln 2} \|\vec{r}_2'(u)\|\,du$

b/c this is something intrinsic about the curve not about its parameterization —

But the speed or velocity does depend on whether we use $\vec{r}_1$ or $\vec{r}_2$.

One natural parameterization:

Arc length function   If we have param. curve $\vec{r}(t)$ $a \le t \le b$

let $s(t) = \int_a^t \|\vec{r}\,'(t')\|\,dt'$

Then, can use $s(t)$ as our parameter!

$0 \le t \le 2\pi$

Ex Helix: $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle \qquad \|\vec{r}\,'(t)\| = \sqrt{2}$

$$s(t) = \int_0^t \|\vec{r}\,'(t')\|\,dt' = \int_0^t \sqrt{2}\,dt' = \sqrt{2}\,t'\Big]_0^t = \sqrt{2}(t-0) = \sqrt{2}\,t$$

So, $s = \sqrt{2}\,t$

$t = s/\sqrt{2}$

New parameterization: $\vec{r}(s) = \left\langle \cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right\rangle \qquad 0 \le s \le 2\pi\sqrt{2}$

In this parameterization, $\vec{r}\,'(s) = \left\langle -\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

$$\|\vec{r}\,'(s)\| = \sqrt{\frac{1}{2}\sin^2\frac{s}{\sqrt{2}} + \frac{1}{2}\cos^2\frac{s}{\sqrt{2}} + \frac{1}{2}} = \sqrt{\frac{1}{2}+\frac{1}{2}} = \underline{1}$$

Indeed this always happens: $s(t) = \int_a^t \|\vec{r}\,'(t)\|\, dt$

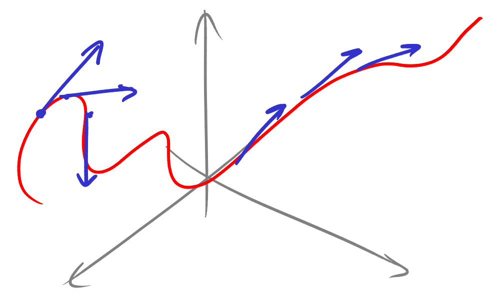
so $\dfrac{ds}{dt} = \|\vec{r}\,'(t)\|$

so $\dfrac{d}{ds}\left[\vec{r}(s)\right] = \dfrac{\frac{d}{dt}\vec{r}(t)}{\frac{ds}{dt}}$ $\left[\dfrac{d}{dt}\vec{r}(s(t)) = \dfrac{d}{ds}\vec{r}(s)\cdot\dfrac{ds}{dt}\right.$ (chain rule) $\left.\right]$

so $\left\|\dfrac{d}{ds}\vec{r}(s)\right\| = \dfrac{\left\|\frac{d}{dt}\vec{r}(t)\right\|}{\left|\frac{ds}{dt}\right|} = \dfrac{\|\vec{r}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \underline{\underline{1}}$

<u>Curvature</u>

Recall unit tangent vector $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$



(If we use arc length param, then $\vec{T}(s) = \vec{r}\,'(s)$, since $\|\vec{r}\,'(s)\| = 1$)

Define $K = \left\|\dfrac{d\vec{T}}{ds}\right\|$ measures how fast $\vec{T}$ is changing

s = arc length param.

Can also compute using another param. t, using $\dfrac{d\vec{T}}{dt} = \dfrac{d\vec{T}}{ds}\cdot\dfrac{ds}{dt}$

so $K = \left\|\dfrac{d\vec{T}/dt}{ds/dt}\right\| = \dfrac{\|d\vec{T}/dt\|}{\|\vec{r}\,'(t)\|}$

Ex What is the curvature K of a circle of radius a?

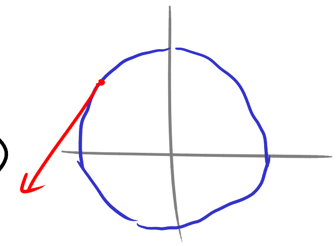
$\vec{r}(t) = \langle h + a\cos t,\ k + a\sin t,\ l \rangle$

circle in plane $z = l$
center (h, k, l)
radius a

$\vec{r}\,'(t) = \langle -a\sin t,\ a\cos t,\ 0 \rangle$

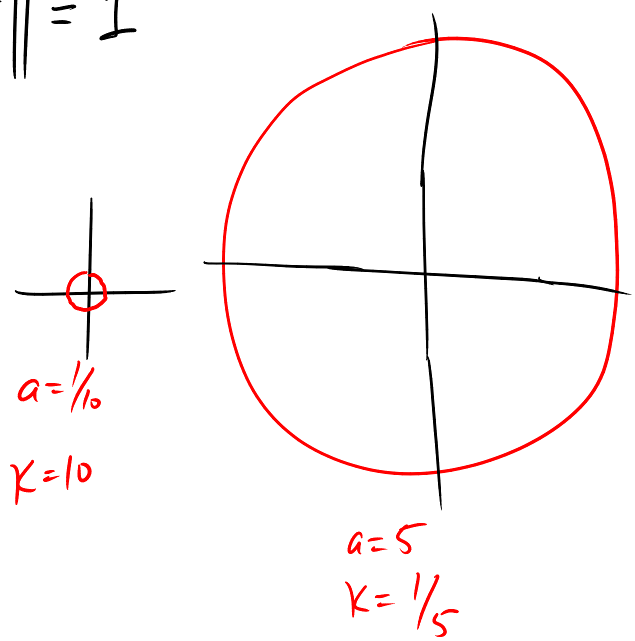
$\|\vec{r}\,'(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a$

$\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \dfrac{\langle -a\sin t, a\cos t, 0\rangle}{a} = \langle -\sin t, \cos t, 0 \rangle$



$\dfrac{d\vec{T}}{dt} = \langle -\cos t, -\sin t, 0 \rangle \qquad \left\| \dfrac{d\vec{T}}{dt} \right\| = 1$

So finally, $K = \dfrac{\left\|\frac{d\vec{T}}{dt}\right\|}{\left\|\frac{d\vec{r}}{dt}\right\|} = \dfrac{1}{a}$



Fact $K(t) = \dfrac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$

Ex Find the curvature of the curve $\vec{r}(t) = \langle t, t^2, t^3 \rangle$

$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$

$\vec{r}\,''(t) = \langle 0, 2, 6t \rangle$

$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \vec{i}\begin{vmatrix} 2t & 3t^2 \\ 2 & 6t \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 3t^2 \\ 0 & 6t \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix}$

$$= \vec{i}\,(12t^2 - 6t^2) - \vec{j}\,6t + \vec{k}\,2$$

$$= \langle 6t^2, -6t, 2 \rangle$$

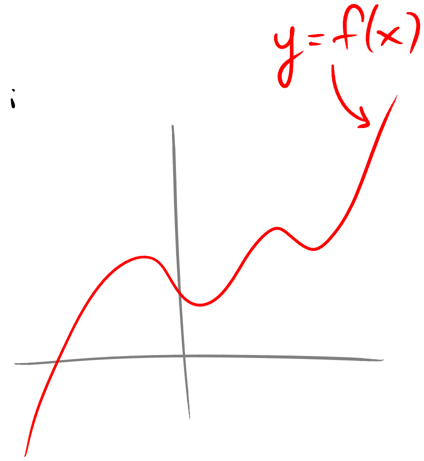
$$\|\vec{r}\,'(t) \times \vec{r}\,''(t)\| = \sqrt{36t^4 + 36t^2 + 4}$$

$$\|\vec{r}\,'(t)\| = \sqrt{1 + 4t^2 + 9t^4}$$

$$\kappa = \frac{\|\vec{r}\,' \times \vec{r}\,''\|}{\|\vec{r}\,'\|^3} = \underline{\underline{\frac{\sqrt{36t^4 + 36t^2 + 4}}{(1 + 4t^2 + 9t^4)^{3/2}}}}$$

e.g. at $t = 0$, $\kappa = \frac{\sqrt{4}}{1^{3/2}} = \underline{\underline{2}}$

Curvature for plane curves given as graphs of functions:



Take $x = t$, then $y = f(t)$

$$\vec{r}(t) = \langle t, f(t), 0 \rangle$$

$$\vec{r}\,'(t) = \langle 1, f'(t), 0 \rangle$$

$$\vec{r}\,''(t) = \langle 0, f''(t), 0 \rangle$$

$$\vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & f'(t) & 0 \\ 0 & f''(t) & 0 \end{vmatrix}$$

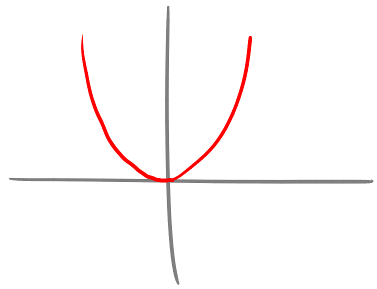
$$= \vec{i} \cdot \begin{vmatrix} f' & 0 \\ f'' & 0 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & f' \\ 0 & f'' \end{vmatrix}$$

$$= \vec{i}\,0 - \vec{j}\,0 + \vec{k}\,f''$$

$$= \langle 0, 0, f'' \rangle$$

$$\kappa = \frac{\|\vec{r}\,' \times \vec{r}\,''\|}{\|\vec{r}\,'\|^3} = \frac{\sqrt{0^2 + 0^2 + (f'')^2}}{(1 + (f')^2)^{3/2}} = \frac{|f''|}{(1 + (f')^2)^{3/2}}$$

Ex Find the curvature of the graph of $y=x^2$ at $(0,0)$, $(1,1)$, $(2,4)$



$$K = \frac{|f''(x)|}{\left(1+(f')^2\right)^{3/2}}$$

$f(x) = x^2$
$f'(x) = 2x$
$f''(x) = 2$

$$K = \frac{2}{\left(1+(2x)^2\right)^{3/2}} = \frac{2}{\left(1+4x^2\right)^{3/2}}$$

at $(0,0)$: $K = \frac{2}{1} = 2$

at $(1,1)$: $K = \frac{2}{5^{3/2}} \approx 0.18$

at $(2,4)$: $K = \frac{2}{17^{3/2}} \approx 0.03$

position $\vec{r}(t)$
velocity $\vec{r}\,'(t)$
accel. $\vec{r}\,''(t)$

NB, K is not a measure of acceleration: