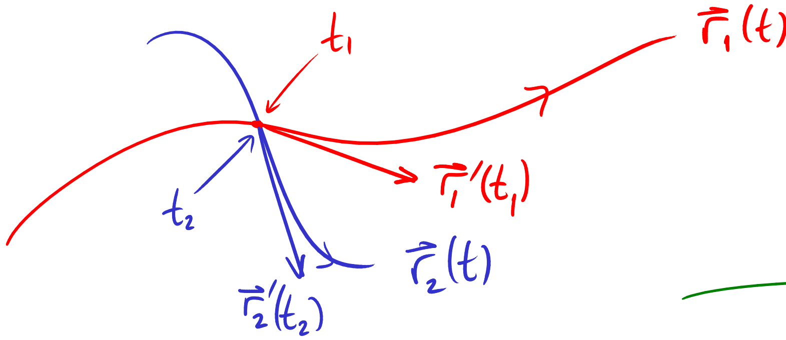
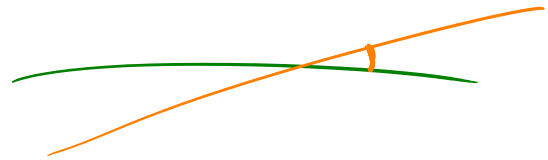
Lecture 12

9 Oct 2014

Remark: given two curves $\vec{r}_1(t)$, $\vec{r}_2(t)$ intersecting at $P = \langle x, y, z\rangle = \vec{r}_2(t_2)$, $P = \vec{r}_1(t_1)$



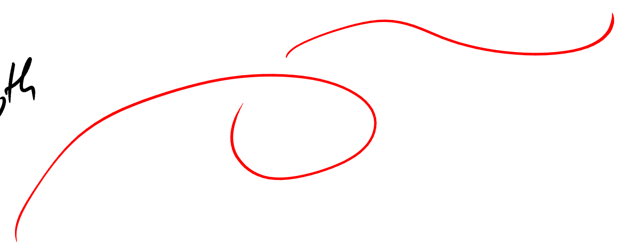


the angle of intersection between the curves at P is the angle between the tangent vectors $\vec{r}_1{}'(t_1)$ and $\vec{r}_2{}'(t_2)$.

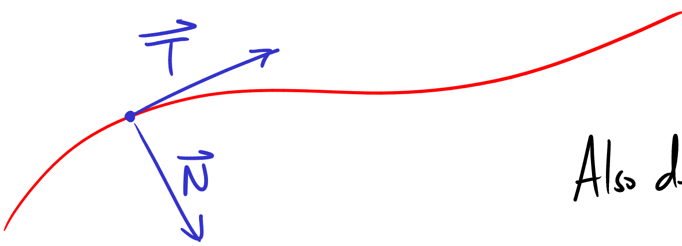
Last time: Curvature of a space curve

$$K = \|\vec{T}'(s)\| \qquad s = \text{arc length param.}$$

$$= \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3} \qquad t = \text{any param.}$$



$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$



Also define: normal vector $\vec{N}(t) = \dfrac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$

Fact: $\vec{N}(t) \perp \vec{T}(t)$.

(as long as $\|\vec{T}'(t)\| \neq 0$, i.e. the curve $\vec{r}(t)$ is bending in some direction)

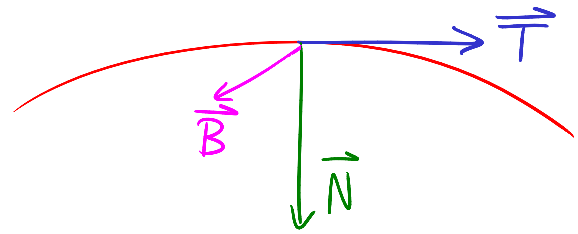
Why? $\vec{T} \cdot \vec{T} = 1$

so $\dfrac{d\vec{T}}{dt} \cdot \vec{T} + \vec{T} \cdot \dfrac{d\vec{T}}{dt} = 0$

$$2\frac{d\vec{T}}{dt}\cdot\vec{T}=0$$

$$\frac{\frac{d\vec{T}}{dt}}{\|\vec{T}'\|}\cdot\vec{T}=0 \quad \text{ie} \quad \vec{N}\cdot\vec{T}=0$$

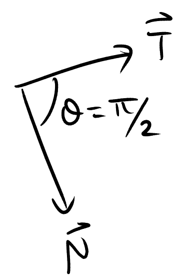
<u>binormal vector</u> $\vec{B}(t)=\vec{T}(t)\times\vec{N}(t)$



($\vec{B}$ points "into" the screen)

All of $\vec{B},\vec{N},\vec{T}$ are unit vectors.

$$\left(\begin{aligned}\|\vec{B}\|=\|\vec{N}\times\vec{T}\|&=\|\vec{N}\|\cdot\|\vec{T}\|\cdot\sin\theta\\&=1\cdot1\cdot1=1\end{aligned}\right)$$

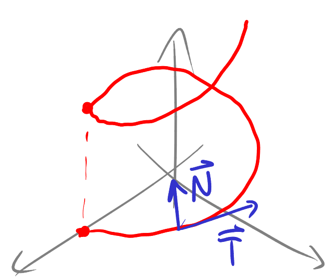


<u>Ex</u> $\vec{r}=\langle\cos t,\sin t,t\rangle$

$$\vec{T}=\left\langle-\frac{\sin t}{\sqrt{2}},\frac{\cos t}{\sqrt{2}},\frac{1}{\sqrt{2}}\right\rangle \qquad \vec{T}'=\left\langle-\frac{\cos t}{\sqrt{2}},-\frac{\sin t}{\sqrt{2}},0\right\rangle$$

$$\vec{N}=\frac{\vec{T}'}{\|\vec{T}'\|}=\frac{\left\langle-\frac{\cos t}{\sqrt{2}},-\frac{\sin t}{\sqrt{2}},0\right\rangle}{\frac{1}{\sqrt{2}}} \qquad \|\vec{T}'\|=\sqrt{\frac{\cos^2 t}{2}+\frac{\sin^2 t}{2}}=\frac{1}{\sqrt{2}}$$

$$=\langle-\cos t,-\sin t,0\rangle$$



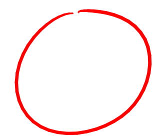
$$\left(\text{check: } \vec{N}\cdot\vec{T}=\left(\frac{\sin t\cos t}{\sqrt{2}}\right)+\left(-\frac{\sin t\cos t}{\sqrt{2}}\right)+0=0\right)$$

$$\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{\sin t}{\sqrt{2}} & \frac{\cos t}{\sqrt{2}} & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \cdots = \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1 \rangle$$

Remark: In addition to the curvature $\kappa(t)$ we could also define torsion $\tau(t)$ by $\dfrac{d\vec{B}}{ds} = -\tau(s)\vec{N}$

(s = arclength param)

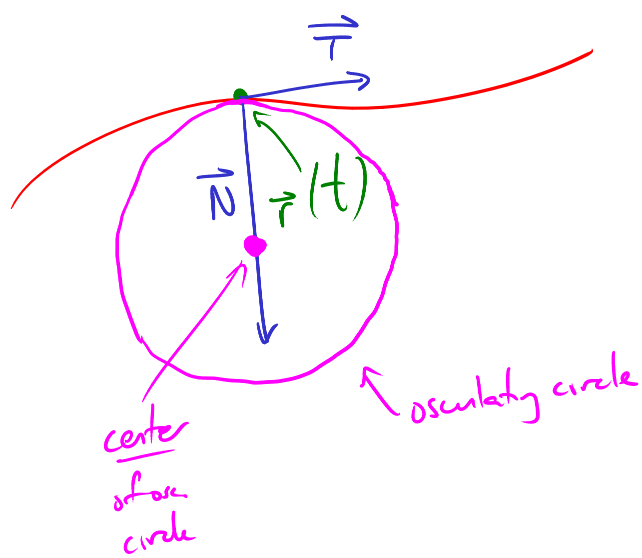
Recall: For a circle of radius r, $\kappa(t) = \frac{1}{r}$ independent of t



Question: Are there lots of other shapes with $\kappa(t)$ indep. of t?

Answer: for any choice of the function $\kappa(t)$ and the function $\tau(t)$ there is a corresponding curve!

Given a curve $\vec{r}(t)$ and a time t we define the osculating plane to the curve at t to be the plane through $\vec{r}(t)$ containing the vectors $\vec{T}(t)$ and $\vec{N}(t)$.



Define the osculating circle to the curve at t to be the circle in the osculating plane, passing through $\vec{r}(t)$, with radius $= \dfrac{1}{\kappa(t)}$, with center on the line thru $\vec{r}(t)$ with direction $\vec{N}$.

Ex For $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

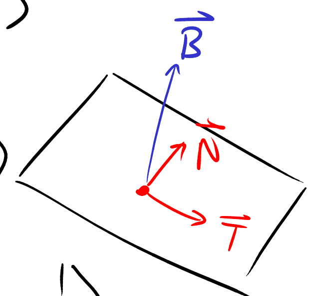
find the osculating plane at $(0, 1, \pi/2)$.

This is $t = \pi/2$.

$$\vec{T} = \left\langle -\frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$$

$$\vec{N} = \langle -\cos t, -\sin t, 0 \rangle = \langle 0, -1, 0 \rangle$$

$$\vec{B} = \vec{T} \times \vec{N} = \left\langle \frac{\sin t}{\sqrt{2}}, -\frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$$



So, we want the plane through $(0, 1, \pi/2)$, $\perp$ to $\left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$

This plane is

$$\left(\vec{r} - \langle 0, 1, \tfrac{\pi}{2} \rangle\right) \cdot \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle = 0$$

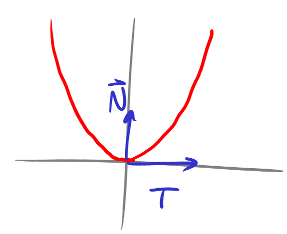
If $\vec{r} = \langle x, y, z \rangle$

$$\langle x, y-1, z-\tfrac{\pi}{2} \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle = 0$$

$$\frac{1}{\sqrt{2}}\left(x + z - \frac{\pi}{2}\right) = 0$$

$$\underline{x + z = \pi/2}$$

Ex Find the osculating circle to the parabola $\begin{array}{l} y = x^2 \\ z = 0 \end{array}$ at $(0, 0, 0)$.



First: osculating plane is just x-y plane.

$$\vec{r}(t) = \langle t, t^2, 0 \rangle$$

$$\vec{r}\,'(t) = \langle 1, 2t, 0 \rangle$$

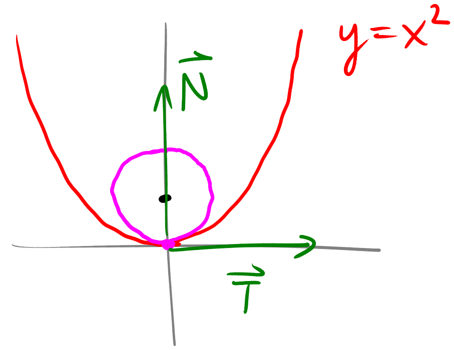
$$\vec{T}(t) = \frac{\langle 1, 2t, 0 \rangle}{\sqrt{1+4t^2}}$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \langle \text{something}, \text{something else}, 0 \rangle$$

<u>Curvature</u>: we calculated last time that $\kappa(0) = 2$ for this curve

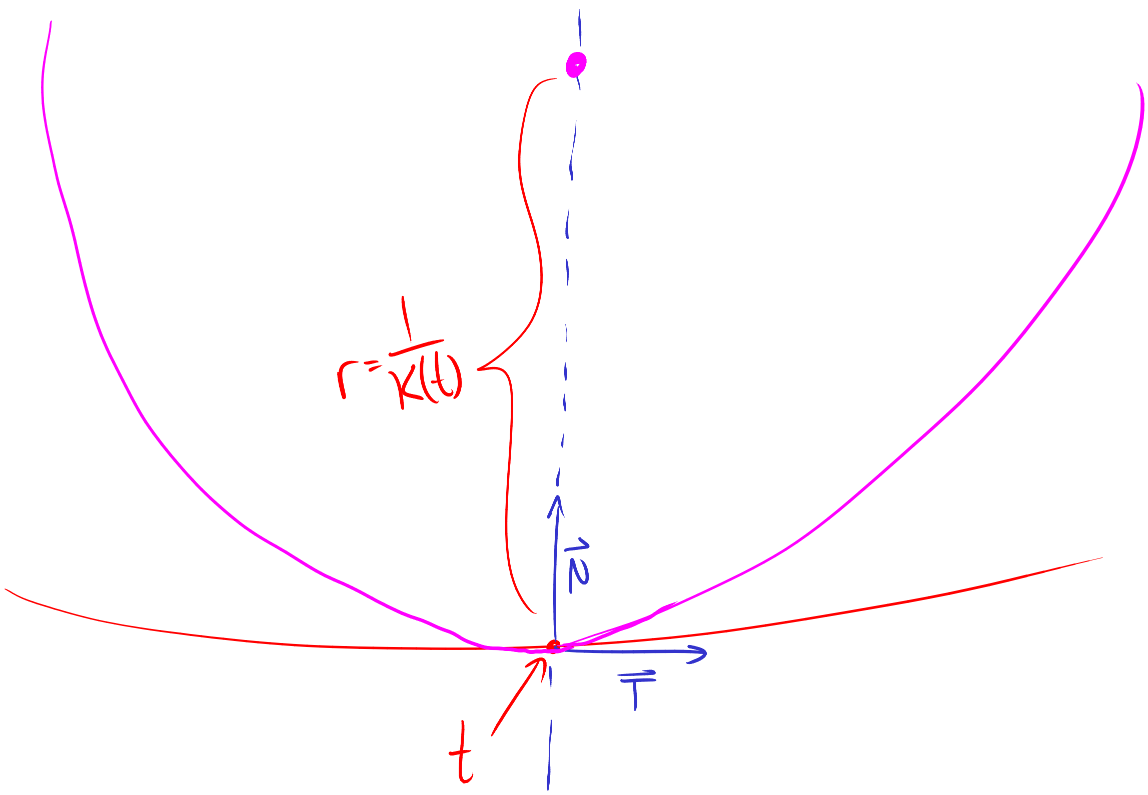
So the osculating circle has radius $\frac{1}{2}$.

Center should lie along the line thru $(0,0,0)$ pointing along $\vec{N}$, ie the y-axis.



$\Longrightarrow$ Center $\left(0, \frac{1}{2}, 0\right)$.

$$\underline{\underline{x^2 + \left(y - \tfrac{1}{2}\right)^2 = 0, \quad z = 0}}$$



Velocity and acceleration (Ch 13.4)

Particle position $\vec{r}(t)$

velocity $\vec{r}'(t) = \vec{v}(t)$

acceleration $\vec{r}''(t) = \vec{a}(t)$

$\vec{a}(t) = \vec{v}'(t)$

Ex A particle is at position $(-1,1,1)$ at $t=0$

with velocity $\langle 1,2,3\rangle$ at $t=0$

with $\vec{a}(t) = \langle 0,0,2\rangle$

Find the velocity $\vec{v}(t)$ and the position $\vec{r}(t)$.

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \int \langle 0,0,2\rangle\,dt = \langle 0,0,2t\rangle + \vec{C}$$

and $\vec{v}(0) = \langle 1,2,3\rangle$ so $\langle 1,2,3\rangle = \langle 0,0,0\rangle + \vec{C}$

ie $\vec{C} = \langle 1,2,3\rangle$

so $\vec{v}(t) = \langle 0,0,2t\rangle + \langle 1,2,3\rangle = \langle 1,2,3+2t\rangle$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \int \langle 1,2,3+2t\rangle\,dt$$

$$= \langle t, 2t, 3t+t^2\rangle + \vec{C}'$$

and $\vec{r}(0) = \langle -1,1,1\rangle$ so

$$\langle -1,1,1\rangle = \langle 0,0,0\rangle + \vec{C}'$$

$$\vec{C}' = \langle -1,1,1\rangle$$

$$\vec{r}(t) = \langle t, 2t, 3t+t^2\rangle + \langle -1,1,1\rangle$$

$= \langle t-1,\ 2t+1,\ 3t+t^2+1 \rangle$

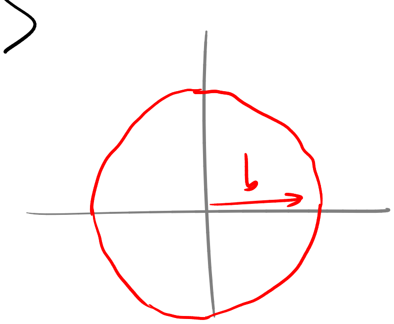
e.g. $\vec{r}(1) = \langle 0, 3, 5 \rangle$

Ex Say $\vec{r}(t) = \langle b\cos\omega t,\ b\sin\omega t,\ 0 \rangle$

b = radius

ω = angular velocity

(particle goes around the circle in time $t = \frac{2\pi}{\omega}$)



$\vec{v}(t) = \langle -b\omega\sin\omega t,\ b\omega\cos\omega t,\ 0 \rangle$

$\|\vec{v}(t)\| = \sqrt{(b\omega)^2(\sin^2\omega t + \cos^2\omega t)} = |b\omega|$

$\vec{a}(t) = \langle -b\omega^2\cos\omega t,\ -b\omega^2\sin\omega t,\ 0 \rangle$

$= b\omega^2 \langle -\cos\omega t,\ -\sin\omega t,\ 0 \rangle$

$\|\vec{a}(t)\| = b\omega^2$