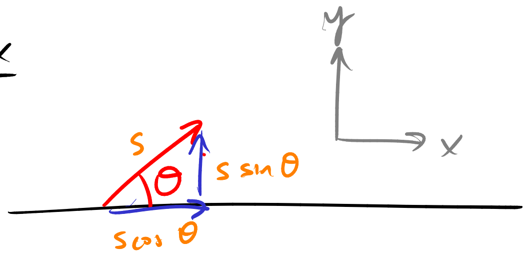
Lecture 13 | 14 Oct 2014

Last time: velocity and acceleration

Ex



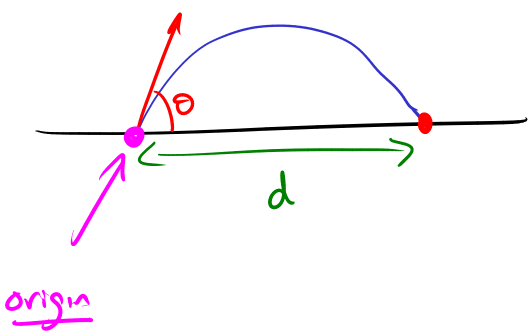
Projectile launched from the ground with fixed speed s, initial angle θ.

Acceleration due only to gravity:

$$\vec{a} = \langle 0, -g \rangle$$

g = gravitational constant ($\approx 9.8\ m/s^2$)

What θ will maximize the distance traveled by the projectile?



$$\vec{a} = \langle 0, -g \rangle$$

$$\vec{v} = \langle 0, -gt \rangle + \vec{C}$$

and $\vec{v}(t=0) = \langle s\cos\theta, s\sin\theta \rangle$

so $\vec{C} = \langle s\cos\theta, s\sin\theta \rangle$

thus $\vec{v} = \langle s\cos\theta,\ s\sin\theta - gt \rangle$

then $\vec{r} = \langle t\cdot s\cos\theta,\ t\cdot s\sin\theta - \frac{1}{2}gt^2 \rangle + \vec{C}'$

and $\vec{r}(t=0) = \langle 0,0 \rangle$

thus $\langle 0,0 \rangle = \langle 0,0 \rangle + \vec{C}'$ so $\vec{C}' = \langle 0,0 \rangle$

and $\vec{r} = \langle t\cdot s\cos\theta,\ t\cdot s\sin\theta - \frac{1}{2}gt^2 \rangle$

When does the projectile hit the ground?

When the y-component of $\vec{r}$ is zero:

$$t\cdot s\sin\theta - \frac{1}{2}gt^2 = 0$$

$$t(s\sin\theta - \frac{1}{2}gt) = 0$$

$t = 0$ <u>or</u> $s\sin\theta - \frac{1}{2}gt = 0$ ie $t = \frac{2s\sin\theta}{g}$

How far does it go? Plug in this value of t to the x-component of $\vec{r}$:

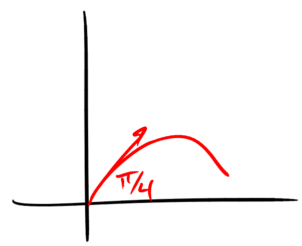
$$d = t \cdot s\cos\theta = \left(\frac{2s\sin\theta}{g}\right)\cdot s\cos\theta = \frac{2s^2\sin\theta\cos\theta}{g}$$

$$= \frac{s^2\sin 2\theta}{g}$$

Critical points of $d(\theta)$: $\frac{d\left(d(\theta)\right)}{d\theta} = \frac{2s^2}{g}\cos 2\theta = 0$

the only crit pt between $\theta = 0$ and $\theta = \pi/2$ is $\underline{\underline{\theta = \pi/4}}$

that gives the max.



Functions of Two Variables (Ch 14.1)

Def A function of two variables is a rule which assigns to any pair of real numbers (x,y) [lying in a set D of pairs, the "domain" of the function] a real number $f(x,y)$.

Ex $f(x,y) = x^2 + y^2x - 1$ $\qquad$ $f(3,2) = 20$

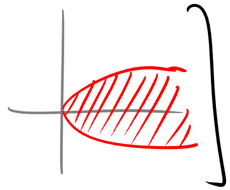
[domain $D = \{\text{all } (x,y)\}$]

$f(x,y) = \sin(xy)$ $\qquad$ $f(3,2) = \sin(6)$

[domain $D = \{\text{all } (x,y)\}$]

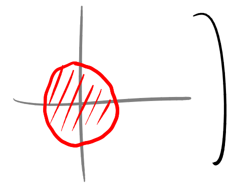
$f(x,y) = x \ln(y^2 - x)$ $\qquad f(3,2) = 0$

[domain $D = \{(x,y) : y^2 - x \geq 0\}$



$f(x,y) = \sqrt{1 - x^2 - y^2}$ $\qquad f(3,2)$ does not exist

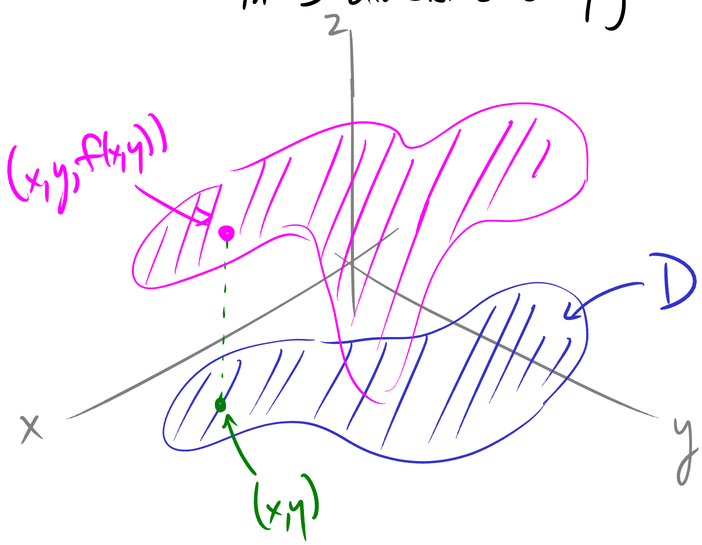
[domain $D = \{(x,y) : 1 - x^2 - y^2 \geq 0\}$



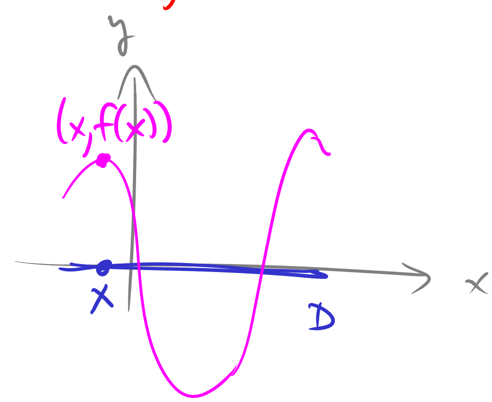
$f(x,y)$ = the # of cars weighing less than x pounds produced in the year y A.D. $\qquad f(3,2) = 0$

[domain $D = \{(x,y) :$ y is a positive integer$\}$]

<u>Graphs</u> The graph of $f(x,y)$ is the set of all points (x,y,z) in 3 dimensions obeying the equation $z = f(x,y)$.



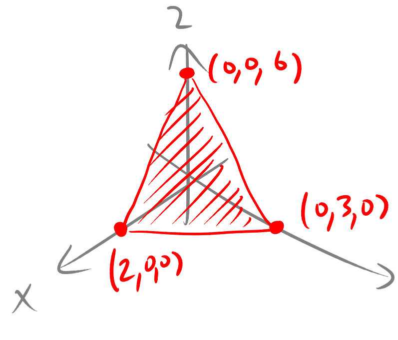
(cf.



<u>Ex</u> Sketch the graph of $f(x,y) = 6 - 3x - 2y$ restricted to the region $x \geq 0, y \geq 0, z \geq 0$ (<u>first octant</u>)

The graph is given by $z = f(x,y)$ i.e. $z = 6 - 3x - 2y$, or $3x + 2y + z = 6$

so it is a plane

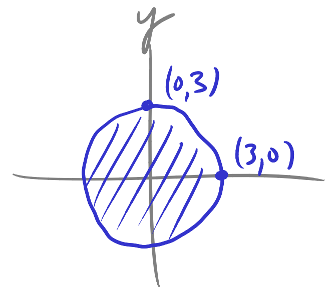


e.g. the maximum value attained by $f(x,y)$ on the domain $\{x \geq 0, y \geq 0\}$ is

$\underline{f(0,0) = 6}$ — highest point of the graph.

<u>Ex</u> Sketch graph of $f(x,y) = \sqrt{9 - x^2 - y^2}$

$D = \{x^2 + y^2 \leq 9\}$

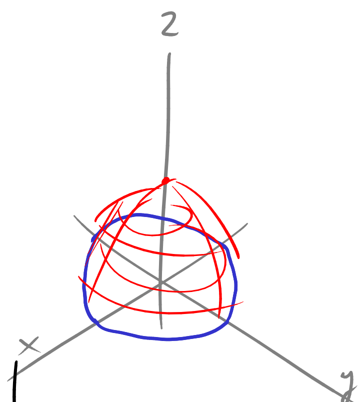


The graph is given by $z = f(x,y)$

$z = \sqrt{9 - x^2 - y^2}$

i.e. $z^2 = 9 - x^2 - y^2$ and $z > 0$

$x^2 + y^2 + z^2 = 9$ and $z > 0$

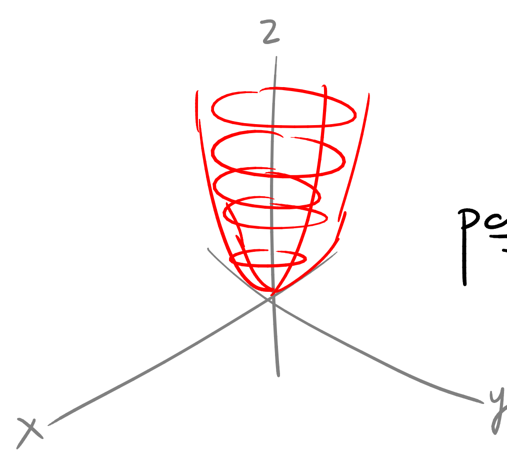


<u>hemisphere</u>

can see here <u>maximum value</u> of $f(x,y)$ is attained at $(x,y) = (0,0)$ $f(x,y) = \sqrt{9 - 0 - 0} = 3$

<u>Ex</u> Sketch graph of $f(x,y) = 4x^2 + y^2$

$z = 4x^2 + y^2$



<u>paraboloid</u>

<u>Traces</u>: in xz-plane, $y=0$, $z = 4x^2$ parabola

in yz-plane, $x=0$, $z = y^2$ parabola

at constant z, $z=k$, $\quad k = 4x^2+y^2$ ellipse

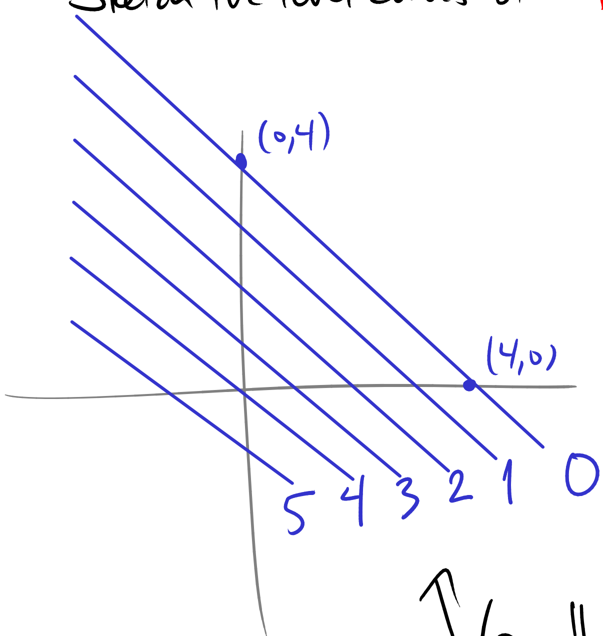
(this $f(x,y)$ does not attain a maximum anywhere

attains minimum at $(x,y)=(0,0)$ $\qquad f(x,y)=0$)

Another way to visualize functions $f(x,y)$:

Level curves $\quad$ The level curves of $f(x,y)$ are the curves $f(x,y)=k$ for fixed k.

Ex $\quad$ Sketch the level curves of $f(x,y) = 4-x-y$.



e.g. $f(x,y)=0$: $\quad 4-x-y=0$

$x+y=4$

$f(x,y)=1$: $\quad 4-x-y=1$

$x+y=3$

$\vdots$

(parallel lines, equally spaced)

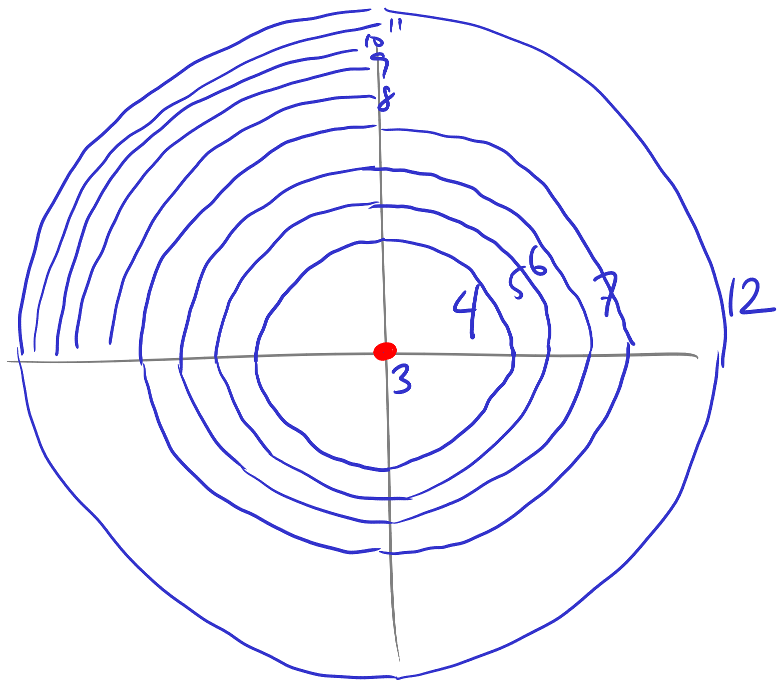
As we walk along a level curve the value of $f(x,y)$ does not change

Ex Sketch level curves of $f(x,y) = x^2+y^2+3$

e.g. $f(x,y)=0$: $x^2+y^2+3=0$

$x^2+y^2=-3$

no such (x,y)



e.g. $f(x,y)=4$: $x^2+y^2+3=4$

$x^2+y^2=1$

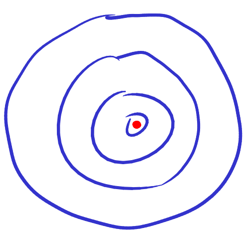
$f(x,y)=7$: $x^2+y^2+3=7$

$x^2+y^2=4$

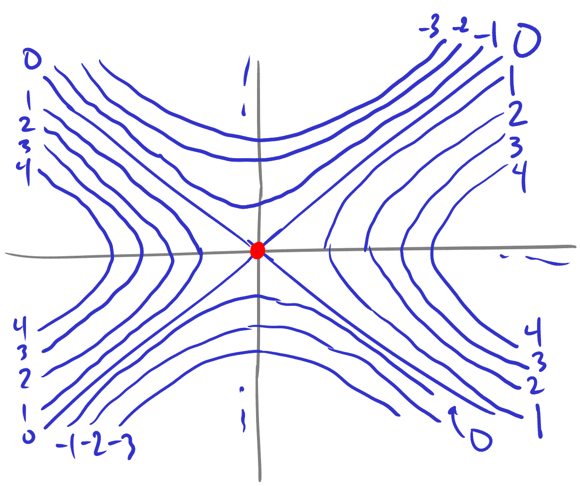
$f(x,y)=12$: $x^2+y^2+3=12$

$x^2+y^2=9$

Spacing of the level curves tells how fast $f(x,y)$ is changing: the more closely spaced they are the faster $f(x,y)$ is changing

A structure like  indicates a local maximum or minimum of $f(x,y)$

Ex Sketch level curves of $f(x,y) = x^2-y^2$



$f(x,y)=0$:

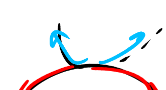
$x^2-y^2=0$

$(x+y)(x-y)=0$

$x=-y$ or $x=y$

$f(x,y)=1$:

$x^2-y^2=1$

the point at $(x,y)=(0,0)$ is a saddle point 

Functions in more than 2 variables:

e.g. $f(x,y,z)$ = density of air at the point (x,y,z) in space

To define the graph of $f(x,y,z)$ we'd need 4 dimensions. $w = f(x,y,z)$

Mathematically this makes perfect sense, but we can't really picture it

But we can still visualize $f(x,y,z)$ using level surfaces:

set $f(x,y,z) = k$ — this gives a surface

e.g. say $f(x,y,z) = x^2+y^2+z^2$

Level surfaces are $k = x^2+y^2+z^2$ spheres of radius $= \sqrt{k}$