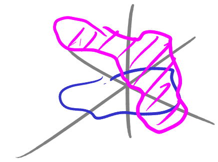
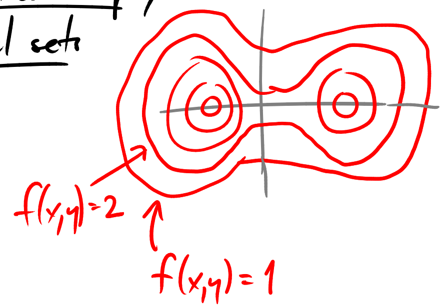
Lecture 14

16 Oct 2014

Last time: functions of multiple variables $f(x,y)$ — their graphs (locus $z = f(x,y)$)



their contour maps / level sets



Limits of Multivariable Functions (Ch 14.3)

Consider the function $f(x,y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2}$.

Not defined at $(x,y)=(0,0)$. (Looks like $f(x,y) = \frac{0}{0}$ there.)

y \ x	-0.5	-0.4	-0.3	-0.2	-0.1	0.	0.1	0.2	0.3	0.4	0.5
-0.5	0.958851	0.972218	0.980844	0.986042	0.988771	0.989616	0.988771	0.986042	0.980844	0.972218	0.958851
-0.4	0.972218	0.983021	0.989616	0.993347	0.99519	0.995739	0.99519	0.993347	0.989616	0.983021	0.972218
-0.3	0.980844	0.989616	0.994609	0.997186	0.998334	0.998651	0.998334	0.997186	0.994609	0.989616	0.980844
-0.2	0.986042	0.993347	0.997186	0.998934	0.999583	0.999733	0.999583	0.998934	0.997186	0.993347	0.986042
-0.1	0.988771	0.99519	0.998334	0.999583	0.999933	0.999983	0.999933	0.999583	0.998334	0.99519	0.988771
0.	0.989616	0.995739	0.998651	0.999733	0.999983	Indeterminate	0.999983	0.999733	0.998651	0.995739	0.989616
0.1	0.988771	0.99519	0.998334	0.999583	0.999933	0.999983	0.999933	0.999583	0.998334	0.99519	0.988771
0.2	0.986042	0.993347	0.997186	0.998934	0.999583	0.999733	0.999583	0.998934	0.997186	0.993347	0.986042
0.3	0.980844	0.989616	0.994609	0.997186	0.998334	0.998651	0.998334	0.997186	0.994609	0.989616	0.980844
0.4	0.972218	0.983021	0.989616	0.993347	0.99519	0.995739	0.99519	0.993347	0.989616	0.983021	0.972218
0.5	0.958851	0.972218	0.980844	0.986042	0.988771	0.989616	0.988771	0.986042	0.980844	0.972218	0.958851

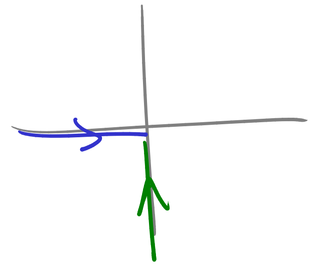
Note that as $(x,y) \to (0,0)$ from any direction, i.e. along any path, the values of $f(x,y)$ are approaching 1.

We will say that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 1$.

Now consider $g(x,y) = \dfrac{x^2 - 2y^2}{x^2 + 2y^2}$. Again, at $(x,y)=(0,0)$ this is not defined — looks like "$g(0,0) = \frac{0}{0}$"

x \ y	-0.5	-0.4	-0.3	-0.2	-0.1	0.	0.1	0.2	0.3	0.4	0.5
-0.5	-0.333333	-0.122807	0.162791	0.515152	0.851852	1.	0.851852	0.515152	0.162791	-0.122807	-0.333333
-0.4	-0.515152	-0.333333	-0.0588235	0.333333	0.777778	1.	0.777778	0.333333	-0.0588235	-0.333333	-0.515152
-0.3	-0.694915	-0.560976	-0.333333	0.0588235	0.636364	1.	0.636364	0.0588235	-0.333333	-0.560976	-0.694915
-0.2	-0.851852	-0.777778	-0.636364	-0.333333	0.333333	1.	0.333333	-0.333333	-0.636364	-0.777778	-0.851852
-0.1	-0.960784	-0.939394	-0.894737	-0.777778	-0.333333	1.	-0.333333	-0.777778	-0.894737	-0.939394	-0.960784
0.	-1.	-1.	-1.	-1.	-1.	Indeterminate	-1.	-1.	-1.	-1.	-1.
0.1	-0.960784	-0.939394	-0.894737	-0.777778	-0.333333	1.	-0.333333	-0.777778	-0.894737	-0.939394	-0.960784
0.2	-0.851852	-0.777778	-0.636364	-0.333333	0.333333	1.	0.333333	-0.333333	-0.636364	-0.777778	-0.851852
0.3	-0.694915	-0.560976	-0.333333	0.0588235	0.636364	1.	0.636364	0.0588235	-0.333333	-0.560976	-0.694915
0.4	-0.515152	-0.333333	-0.0588235	0.333333	0.777778	1.	0.777778	0.333333	-0.0588235	-0.333333	-0.515152
0.5	-0.333333	-0.122807	0.162791	0.515152	0.851852	1.	0.851852	0.515152	0.162791	-0.122807	-0.333333

In this case, as $(x,y) \to (0,0)$ the behavior of $g(x,y)$ depends on <u>which direction</u> (x,y) is coming from, i.e. on which path we take into $(0,0)$.



along path $y=0$: $\quad g(x,y) = \frac{x^2}{x^2} = 1$

along path $x=0$: $\quad g(x,y) = \frac{-2y^2}{2y^2} = -1$

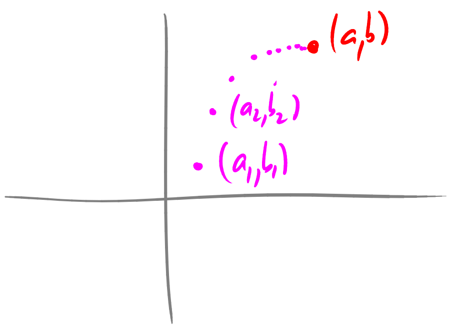
In this situation we'll say $\lim_{(x,y)\to(0,0)} g(x,y)$ does <u>not</u> exist.

<u>Def</u> Given a function $f(x,y)$

we say $\lim_{(x,y)\to(a,b)} f(x,y) = L$ if

for every sequence (a_n,b_n) with $\lim_{n\to\infty} a_n = a \quad \lim_{n\to\infty} b_n = b$

$(a_n,b_n) \neq (a,b)$



we have $\lim_{n\to\infty} f(a_n,b_n) = L$.

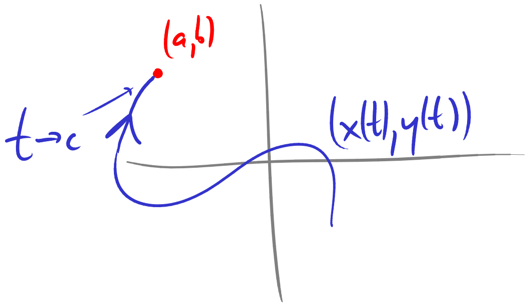
If there is <u>no</u> L for which $\lim_{(x,y)\to(a,b)} f(x,y) = L$, we say $\lim_{(x,y)\to(a,b)} f(x,y)$ <u>does not exist</u>.

<u>Ex</u> $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1$, $\quad \lim_{(x,y)\to(0,0)} \frac{x^2-2y^2}{x^2+2y^2}$ does not exist.

<u>Fact</u> <u>If</u> $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and P is any path going into (a,b)

i.e. $P = (x(t), y(t))$ and $\lim_{t\to c} x(t) = a$

$\lim_{t\to c} y(t) = b$

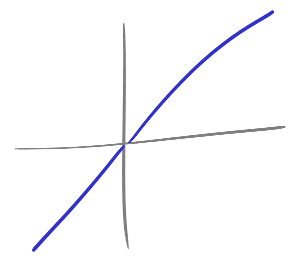


<u>then</u> $\lim\limits_{t\to c} f(x(t), y(t)) = L$.

<u>Ex</u> We said $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2} = 1$. $\qquad f(x,y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2}$

Let's <u>test</u> that: consider the path $(x(t), y(t)) = (t,t)$

As $t\to 0$, $\lim\limits_{t\to 0} (x(t), y(t)) = (0,0)$.



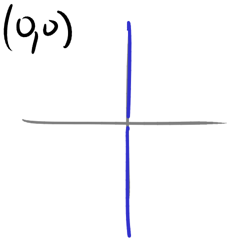
$$\lim_{t\to 0} f(x(t), y(t)) = \lim_{t\to 0} \frac{\sin(t^2+t^2)}{t^2+t^2}$$

$$= \lim_{t\to 0} \frac{\sin(2t^2)}{2t^2} = \lim_{t\to 0} \frac{4t\cos(2t^2)}{4t} = \lim_{t\to 0} \cos(2t^2)$$

$$= \underline{\underline{1}}$$

Could also take the path $(x(t), y(t)) = (0,t)$ $\qquad \lim\limits_{t\to 0} (x,y) = (0,0)$

$$\lim_{t\to 0} f(x(t), y(t)) = \lim_{t\to 0} \frac{\sin(t^2)}{t^2} = \cdots = \underline{\underline{1}}$$



<u>Ex</u> If $f(x,y) = \dfrac{xy}{x^2+y^2}$ does $\lim\limits_{(x,y)\to(0,0)}$ exist?

Limit along the path $(x(t), y(t)) = (t,0)$: $\lim\limits_{t\to 0} \dfrac{x(t)\,y(t)}{x(t)^2+y(t)^2} = \lim\limits_{t\to 0} \dfrac{t\cdot 0}{t^2+0^2} = \underline{\underline{0}}$

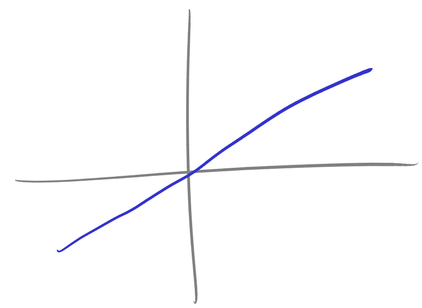
along $(x(t), y(t)) = (0,t)$: $\lim\limits_{t\to 0} \dfrac{x(t)\,y(t)}{x(t)^2+y(t)^2} = \lim\limits_{t\to 0} \dfrac{0\cdot t}{0^2+t^2} = \underline{\underline{0}}$

along $(x(t),y(t)) = (t,t)$: $\displaystyle\lim_{t\to 0} \frac{x(t)y(t)}{x(t)^2+y(t)^2} = \lim_{t\to 0} \frac{t\cdot t}{t^2+t^2} = \underline{\underline{\tfrac{1}{2}}}$

Different limits along different paths!

So $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

Ex If $f(x,y) = \dfrac{xy^2}{x^2+y^4}$



any line through $(0,0)$ (except the y-axis) can be written $y = mx$ — param as $(x(t),y(t)) = (t, mt)$

Try taking the limit along such a line:

$$\lim_{t\to 0} f(x(t),y(t)) = \lim_{t\to 0} \frac{x(t)\,y(t)^2}{x(t)^2+y(t)^4} = \lim_{t\to 0} \frac{t\cdot(mt)^2}{t^2+(mt)^4}$$

$$= \lim_{t\to 0} \frac{m^2t^3}{t^2+m^4t^4}$$

$$= \lim_{t\to 0} \frac{m^2 t}{1+m^4t^2} = 0$$

But, now try taking limit along a parabola, $y^2 = x$.

$(x(t),y(t)) = (t^2,t)$

$$\lim_{t\to 0} f(x(t),y(t)) = \lim_{t\to 0} \frac{t^2\cdot t^2}{(t^2)^2+t^4} = \lim_{t\to 0} \frac{t^4}{2t^4} = \frac{1}{2}$$

So, $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist!

Some good news:

Fact

$$\lim_{(x,y)\to(a,b)} c = c$$

$$\lim_{(x,y)\to(a,b)} x = a$$

$$\lim_{(x,y)\to(a,b)} y = b$$

Limit Laws hold for these 2-d limits just as for 1-d ones e.g.

$$\lim_{(x,y)\to(a,b)} f(x,y)+g(x,y) = \lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y)$$

$$\lim f\cdot g = \left(\lim f\right)\cdot\left(\lim g\right)$$

$$\lim \frac{f}{g} = \frac{\lim f}{\lim g}$$

$\left[\text{if } \lim_{(x,y)\to(a,b)} g \neq 0\right]$

provided that all the limits on the right side exist

Ex

$$\lim_{(x,y)\to(1,2)} \frac{x^2y}{x+y} = \frac{\lim_{(x,y)\to(1,2)} x^2y}{\lim_{(x,y)\to(1,2)} x+y}$$

$$= \frac{\left(\lim_{(x,y)\to(1,2)} x\right)^2\left(\lim_{(x,y)\to(1,2)} y\right)}{\left(\lim_{(x,y)\to(1,2)} x\right)+\left(\lim_{(x,y)\to(1,2)} y\right)}$$

$$= \frac{1^2\,2}{1+2} = \frac{2}{3}$$

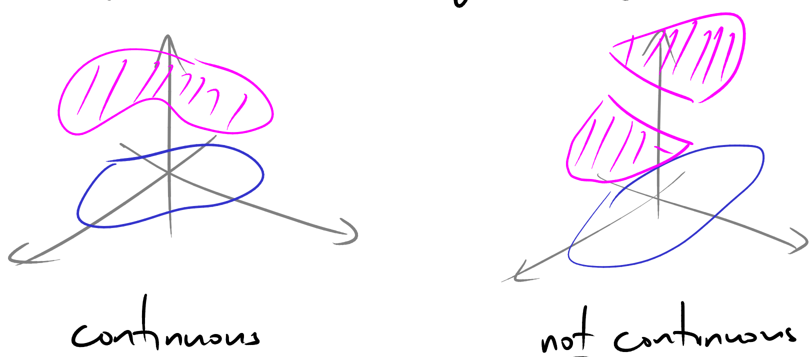
Continuity

Def We say $f(x,y)$ is continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

We say $f(x,y)$ is continuous on D (domain) if $f(x,y)$ is continuous at every point of D.

(Intuitively: f is continuous if its graph is obtained by deforming D, without cutting)



Ex $f(x,y) = c \quad f(x,y) = x \quad f(x,y) = y$ are all continuous functions.

Fact Sums, products, quotients of continuous functions are continuous.

Ex Polynomials in (x,y) are continuous. e.g. $f(x,y) = 4x^2 + 6xy - y^3$

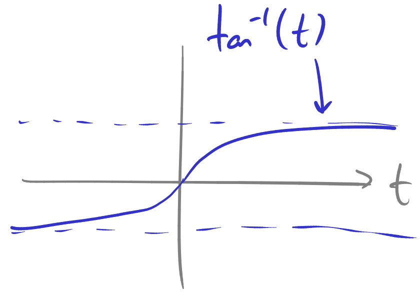
Rational functions of (x,y) are continuous (on their domain) e.g. $f(x,y) = \dfrac{x^2+y^3}{x-y}$ is continuous on $D = \{(x,y) : x \neq y\}$

Fact If $f(x,y)$ is continuous at (a,b) and $g(t)$ is continuous at $f(a,b)$

then $g(f(x,y))$ is continuous at (a,b)

<u>Ex</u> $f(x,y) = \sin\left[\frac{x^2+y^3}{x-y}\right]$ is continuous on $D = \{(x,y) : x \neq y\}$

<u>Ex</u> $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$



is continuous on

$D = \{(x,y) : x \neq 0\}$

<u>Partial Derivatives</u> (Ch 14.3)

Given $f(x,y)$ we define

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

("treat y as a <u>constant</u> but x as a <u>variable</u>")

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

("treat x as a <u>constant</u> and y as a <u>variable</u>")

<u>Notation</u>:

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x}$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y}$$

To compute f_x: treat y as const

<u>Ex</u> If $f(x,y) = \sin(xy) + x^2 + y^3$

then $f_x(x,y) = y\cos(xy) + 2x$

$f_y(x,y) = x\cos(xy) + 3y^2$