Lecture 15

21 Oct 2014

Ex $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} + 2 = ?$

Try plugging in: $\frac{0}{0} + 2$ ✗

Try taking limit along lines: $(x,y) = (at, bt)$

then as $t \to 0$, $(x,y) \to (0,0)$

we consider $\displaystyle\lim_{t\to 0} \frac{3(at)^2(bt)}{(at)^2+(bt)^2} + 2 = \lim_{t\to 0} \frac{3a^2b\,t^3}{(a^2+b^2)t^2} + 2$

$$= \lim_{t\to 0} \frac{3a^2b}{(a^2+b^2)} \cdot t + 2$$

$$= 2$$

So, might suspect that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} + 2 = 2$

How to prove it?

Fact (Squeeze Thm): If $|g(x,y)| \le f(x,y)$ and $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = 0$

then $\displaystyle\lim_{(x,y)\to(a,b)} g(x,y) = 0$.

Fact: If $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = 0$ then $\displaystyle\lim_{(x,y)\to(a,b)} |f(x,y)| = 0$.

We'll use this here: $\left|\frac{x^2}{x^2+y^2}\right| \le 1$, so $\left|\frac{x^2}{x^2+y^2} \cdot 3y\right| \le 3|y|$

And we know $\displaystyle\lim_{(x,y)\to(0,0)} 3y = 0$. So $\displaystyle\lim_{(x,y)\to(0,0)} 3|y| = 0$

and thus by Squeeze Thm $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}\cdot 3y = 0$

so $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} + 2 = \underline{\underline{2}}$.

Ex $\frac{d}{dy}\left[\int_3^y \tan\left(\frac{x^9+27}{x^7+\sqrt{x^3+1}}\right) dx\right] = ?$

(FTC)

$= \tan\left(\frac{y^9+27}{y^7+\sqrt{y^3+1}}\right)$

If $f(x,y) = \int_x^y \cos(t^2+4)\, dt$

$\frac{\partial f}{\partial y} = \cos(y^2+4)$ $\qquad \frac{\partial f}{\partial x} = -\cos(x^2+4)$

More about Partial Derivatives (Ch 14.n)

Recall we defined last time (for $f(x,y)$)

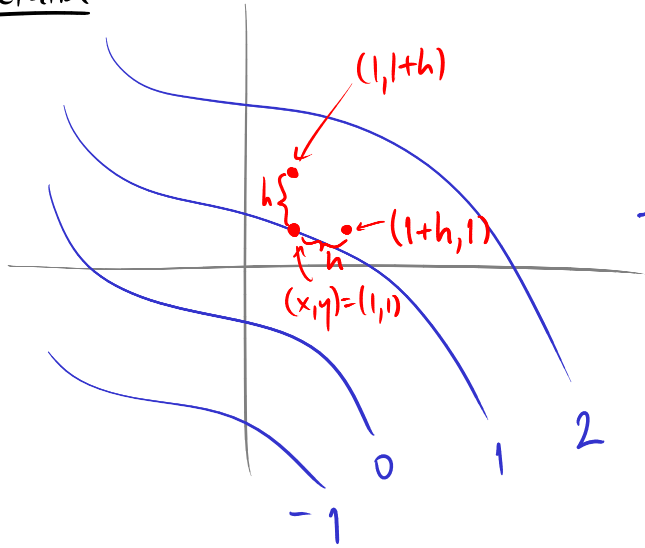
$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$

$f_y = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$

Ex $f(x,y) = x^2 \sin(cy^2)$

has $f_x = 2x \sin(cy^2)$ $\qquad f_y = x^2 \cdot 2cy \cos(cy^2)$

Interpretation:



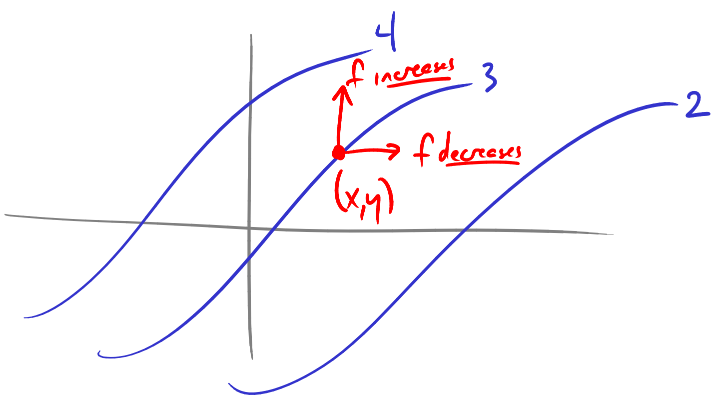
$$f(1,1)=1$$

$$f_x(1,1)=\lim_{h\to 0}\frac{f(1+h,1)-f(1,1)}{h}>0$$

(f increases as we move to right i.e. increase x)

$$f_y(1,1)=\lim_{h\to 0}\frac{f(1,1+h)-f(1,1)}{h}>0$$

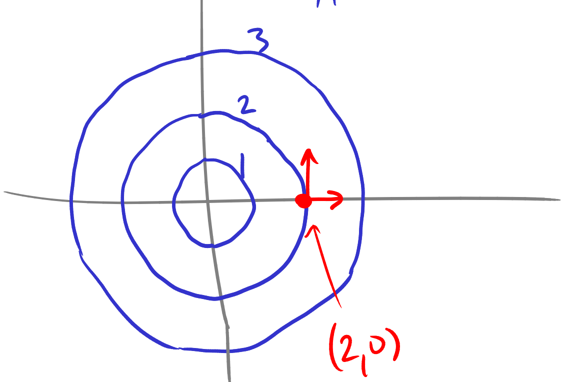
similarly



$$f_x(x,y)<0$$

$$f_y(x,y)>0$$

$$f(x,y)=\sqrt{x^2+y^2}$$



$$f_x(2,0)>0$$

$f_y(2,0)=0$ (moving in y-direction moves tangent to contour-line)

$$f_y(x,y)=\frac{y}{(x^2+y^2)^{1/2}}$$

So indeed $f_y(2,0)=\frac{0}{2}=0$

But, $f_{yy}(x,y)=\frac{\partial}{\partial y}\frac{y}{(x^2+y^2)^{1/2}}$

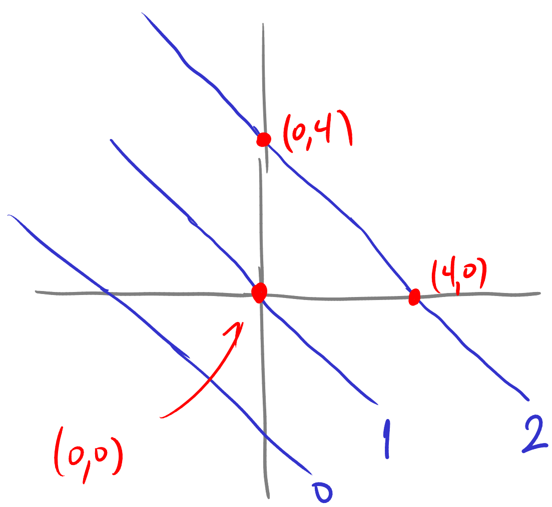
$$= \frac{\partial}{\partial y}\, y \cdot (x^2+y^2)^{-1/2}$$

$$= (x^2+y^2)^{-1/2} + y \cdot 2y \cdot (-\tfrac{1}{2}) \cdot (x^2+y^2)^{-3/2}$$

$$= \frac{1}{(x^2+y^2)^{1/2}} - \frac{y^2}{(x^2+y^2)^{3/2}}$$

so $f_{yy}(2,0) = \frac{1}{2}$ — i.e. if we look at the function $f(2,y)$ we have a <u>local minimum</u> at $y=0$

Ex



Estimate $f_x(0,0)$.

As x increases by 4 units, f increases by 1 unit.

So $f_x \approx \frac{1}{4}$.

i.e. $f_x \approx \dfrac{f(0+h,0) - f(0,0)}{h}$

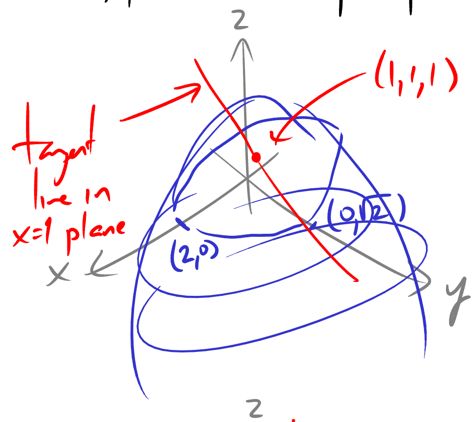
we plug in $h=4$,

$$f_x \approx \frac{f(4,0) - f(0,0)}{4} = \frac{2-1}{4} = \frac{1}{4}$$

[Similarly $f_y \approx \frac{1}{4}$.]

Rk $f_x(a,b)$ is the slope of the tangent line to the graph $z = f(x,y)$ at $(a, b, f(a,b))$ lying in the plane $y=b$.

Ex Say $f(x,y) = 4 - x^2 - 2y^2$. $z = f(x,y)$ is an elliptic paraboloid



Let's look at its tangent lines at the point $(x,y,z) = (1,1,1)$

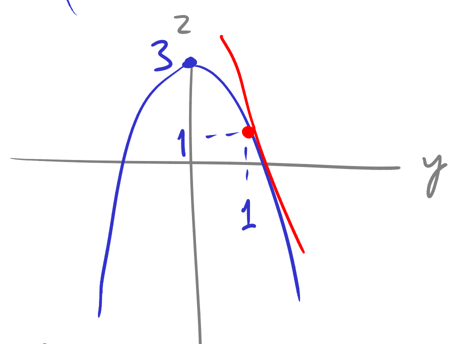
e.g. tangent line in the plane $x=1$:

$$\text{slope} = f_y(x,y)$$

$$= -4y$$

plug in $y=1$: slope $= -4$

So the line is
$$\begin{cases} z = 5 - 4y \\ x = 1 \end{cases}$$



$$\left[\begin{aligned} z = f(1,y) &= 4 - 1 - 2y^2 \\ &= 3 - 2y^2 \end{aligned}\right]$$

Implicit differentiation for partial derivatives:

Suppose a function $z(x,y)$ is given implicitly by an equation,

e.g.
$$x^3 + y^3 + z^3 + 6xyz = 1$$

and we want to compute $\frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$ without explicitly solving for z first.

(e.g. to find slopes of tangent lines to this surface)

The trick: if we want $\frac{\partial z}{\partial x}$, apply $\frac{\partial}{\partial x}$ to the whole equation — (holding y constant)

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

$$(3z^2+6xy)\frac{\partial z}{\partial x} = -3x^2-6yz$$

$$\frac{\partial z}{\partial x} = \frac{-(3x^2+6yz)}{3z^2+6xy}$$

e.g. at $(x,y,z)=(0,0,1)$

we get $\frac{\partial z}{\partial x} = -\frac{0+0}{3+0} = \underline{\underline{0}}$

Could do similarly for $\frac{\partial z}{\partial y}$: apply $\frac{\partial}{\partial y}$ to whole equation

this would give

$$\frac{\partial z}{\partial y} = -\frac{y^2+2xz}{z^2+2xy}$$

<u>Higher derivatives</u>

There are various notions of "second partial derivative"

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \qquad f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

<u>Ex</u> Say $f(x,y) = e^{xy}\sin y$

$$f_x = ye^{xy}\sin y \qquad f_y = xe^{xy}\sin y + e^{xy}\cos y$$

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = y^2e^{xy}\sin y$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y) = x^2e^{xy}\sin y + xe^{xy}\cos y + xe^{xy}\cos y - e^{xy}\sin y = (x^2-1)e^{xy}\sin y + 2xe^{xy}\cos y$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = e^{xy}\sin y + xye^{xy}\sin y + ye^{xy}\cos y$$

$$f_{yx} = \frac{\partial}{\partial x}(f_y) = e^{xy}\sin y + xye^{xy}\sin y + ye^{xy}\cos y$$

Note $f_{xy} = f_{yx}$!

$$\begin{tikzcd}
& & & f \arrow[dll] \arrow[drr] & & & \\
& f_x \arrow[dl] \arrow[dr] & & & & f_y \arrow[dl] \arrow[dr] & \\
f_{xx} & & f_{xy} \arrow[rr, phantom, "="] & & f_{yx} & & f_{yy}
\end{tikzcd}$$

Fact: If f_{xy} and f_{yx} both exist and are continuous then $f_{xy} = f_{yx}$.

More generally, e.g.

$$f_{xyyxx} = f_{xxyxy}$$

order doesn't matter