Lecture 16

23 Oct 2014

Midterm 2 Nov 4 in class — email me if this conflicts with your registration time

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Last time: partial derivatives, higher partial derivatives

e.g.

$$f(x,y) = xy^2$$

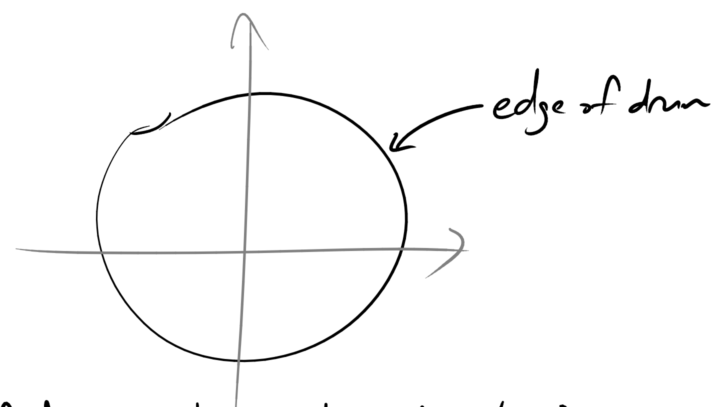
$$f_x = y^2 \qquad f_y = 2xy$$

$$f_{xx} = 0 \qquad f_{xy} = 2y \qquad f_{yx} = 2y \qquad f_{yy} = 2x$$

$$\underline{f_{xy} = f_{yx}} \quad \text{[if both continuous]}$$

One application of these:

waves on a drum



there's a function

$z(x,y,t)$ = height of drum membrane at position (x,y) and time t

its behavior is governed by "wave equation"

$$a^2 \frac{\partial^2 z}{\partial t^2} = \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right) \qquad (*)$$

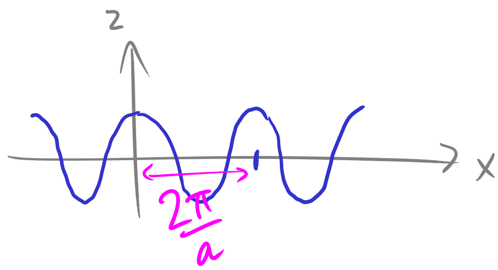
one solution of this equation: $z(x,y,t) = \cos(t-ax)$

check: $\frac{\partial z}{\partial x} = a\sin(t-ax)$ $\quad \frac{\partial z}{\partial y} = 0$ $\quad \frac{\partial z}{\partial t} = -\sin(t-ax)$

$\frac{\partial^2 z}{\partial x^2} = -a^2\cos(t-ax)$ $\quad \frac{\partial^2 z}{\partial y^2} = 0$ $\quad \frac{\partial^2 z}{\partial t^2} = -\cos(t-ax)$

plugging into $(*)$: $-a^2\cos(t-ax) = -a^2\cos(t-ax)$ ✓

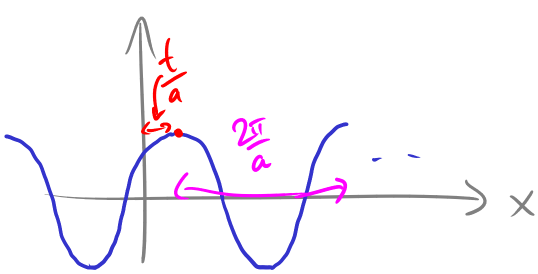
How to picture this solution: at fixed time $t=0$, $z(x,y,t=0) = \cos(-ax)$



as we vary t,

$$z(x,y,t) = \cos(t-ax)$$
$$= \cos\left(-a\left(x-\frac{t}{a}\right)\right)$$

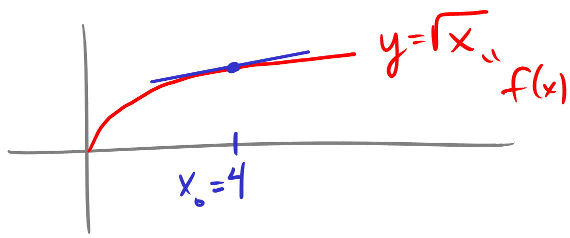
$\underbrace{\hphantom{xx}}_{}$ x gets shifted by $\frac{t}{a}$



(so wave is moving to the right with speed $1/a$)

Tangent planes and linear approximation (Ch 14.4)

Recall from 1-variable calculus: How to estimate $\sqrt{4.1}$?



Consider the function $f(x) = \sqrt{x}$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Set $x_0 = 4$

$f(x_0) = \sqrt{4} = 2 \qquad f'(x_0) = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$

Tangent line to $y = f(x)$ at $x = x_0$:

$$y = f(x_0) + (x - x_0) f'(x_0)$$

Here, plug in $x = 4.1$: tangent line

$$y = 2 + (4.1 - 4)\,\frac{1}{4}$$
$$= 2 + \frac{1}{40} = 2.025$$

Since x is close to x_0, $f(x)$ is well approx. by the tangent line:

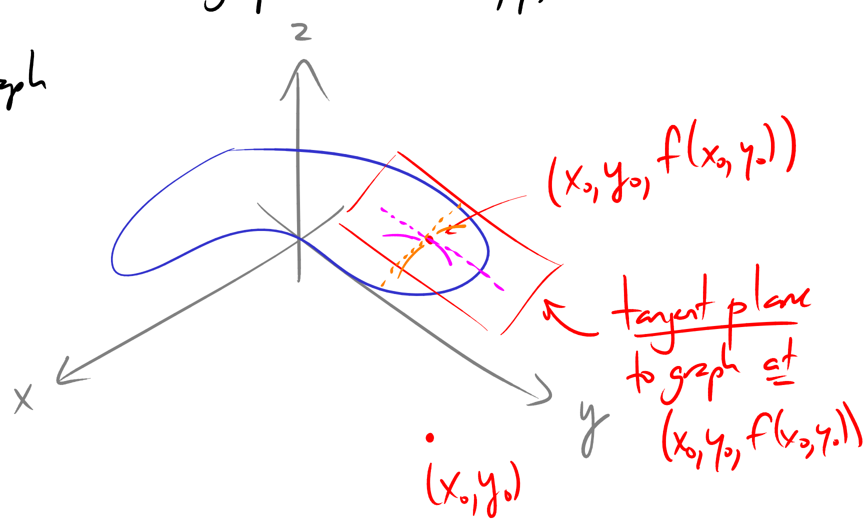
$$f(x) \approx f(x_0) + (x - x_0) f'(x_0)$$

i.e. $\sqrt{4.1} \approx 2.025$

(actually $\sqrt{4.1} = 2.024846\ldots$)

Now suppose we want to do the same for a function $f(x,y)$.

Near any (x_0, y_0) we may approximate the graph $z = f(x,y)$ by a <u>tangent plane</u>, tangent to graph at $(x_0, y_0, f(x_0, y_0))$



(What does "tangent plane" mean? If we look at any path which lies on the graph, the <u>tangent line to that path</u> lies in the tangent plane)

Tangent plane at $(x_0, y_0, f(x_0,y_0))$ is given by the equation

$$z = f(x_0,y_0) + (x-x_0) f_x(x_0,y_0) + (y-y_0) f_y(x_0,y_0)$$

[an approximation to

$$z = f(x,y)$$]

Why? The tangent plane is a plane through $(x_0, y_0, f(x_0,y_0))$ so it must be of the form

$$A(x-x_0) + B(y-y_0) + C(z - f(x_0,y_0)) = 0$$

Divide by C:

$$A/C\,(x-x_0) + \frac{B}{C}(y-y_0) + z - f(x_0,y_0) = 0$$

$$z = f(x_0,y_0) - \frac{A}{C}(x-x_0) - \frac{B}{C}(y-y_0)$$

ie $$z = f(x_0,y_0) + a(x-x_0) + b(y-y_0) \quad (*)$$

$a = -\frac{A}{C}$

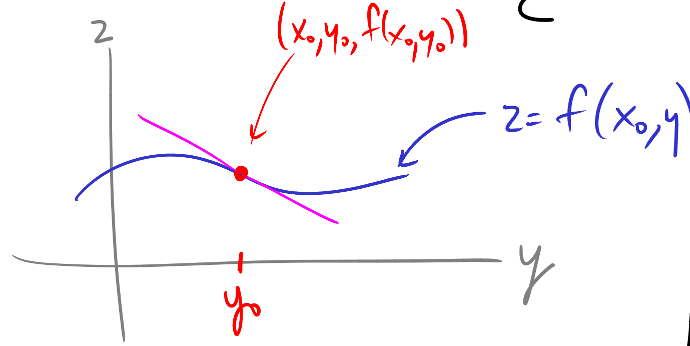
$b = -\frac{B}{C}$

Need to determine a, b.

Slice by the plane $x = x_0$:

here we know <u>tangent line</u> is given by

$$z = f(x_0,y_0) + (y-y_0) f_y(x_0,y_0)$$



But substituting $x = x_0$ in $(*)$, we get

$$z = f(x_0,y_0) + (y-y_0)\cdot b$$

They match <u>only if</u> we put $b = f_y(x_0,y_0)$

Similarly, looking at $y = y_0$, get $a = f_x(x_0,y_0)$

Ex Find the tangent plane to the graph $z = \underbrace{2x^2+y^2}_{f(x,y)}$ at $(x,y,z) = (1,1,3)$,

use it to estimate $f(1.1, 0.95)$.

$x_0 = 1$
$y_0 = 1$

$f(x,y) = 2x^2 + y^2$ $\quad f(1,1) = 3$

$f_x = 4x$ $\quad f_x(1,1) = 4$

$f_y = 2y$ $\quad f_y(1,1) = 2$

Tangent plane: $z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$z = 3 + 4(x-1) + 2(y-1)$

$z = -3 + 4x + 2y$

estimate: $f(1.1, 0.95)$

$x = 1.1$
$y = 0.95$

$$f(x,y) \approx -3 + 4(1.1) + 2(0.95)$$
$$= -3 + 4.4 + 1.9$$
$$= 3.3$$

(exact answer: $f(x,y) = 2(1.1)^2 + (0.95)^2 = 3.3225$)

But, as (x,y) goes further from (x_0,y_0) the accuracy gets worse:

e.g. at $(x,y) = (2,2)$ $\quad f(2,2) = 2(2^2) + 2^2 = 12$

our estimate $f(2,2) \approx -3 + 4(2) + 2(2) = 9$

Ex Estimate $(1.1)^3(0.9)^4$ $\qquad (x_0,y_0)=(1,1)$

Let $f(x,y)=x^3y^4$ $\qquad f(1,1)=1$

$f_x=3x^2y^4$ $\qquad f_x(1,1)=3$

$f_y=4x^3y^3$ $\qquad f_y(1,1)=4$

$$\begin{aligned} f(x,y) &\approx f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0) \\ &= 1+3(x-1)+4(y-1) \\ &= 1+3(0.1)+4(-0.1) \\ &= 0.9 \end{aligned}$$

exact answer: $f(1.1,0.9)=(1.1)^3(0.9)^4=0.8733\ldots$

Terminology

We call $L(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$

the <u>linearization</u> of f at (x_0,y_0)

So, the tangent plane to $z=f(x,y)$ at $(x_0,y_0,f(x_0,y_0))$ is the graph of the <u>linearization</u> of f at (x_0,y_0)

$$z=L(x,y).$$

Notation Another convenient way of thinking about linearization: <u>total differential</u>

Given $f = f(x,y)$

define
$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

resulting small change in $f(x,y)$

"small change in x"

"small change in y"

Linear approx says: write
$$\Delta x = x - x_0$$
$$\Delta y = y - y_0$$
$$\Delta f = f(x,y) - f(x_0,y_0)$$

if $\Delta x, \Delta y$ are small, can replace
$$dx \to \Delta x$$
$$dy \to \Delta y$$
$$df \to \Delta f$$

and get an approximately true equation,
$$\Delta f \approx \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y$$

Ex $f(x,y) = x^2 + 3xy - y^2$
$$df = (2x\,dx) + (3y\,dx + 3x\,dy) - 2y\,dy$$
$$= \underbrace{(2x+3y)}_{\frac{\partial f}{\partial x}} dx + \underbrace{(3x-2y)}_{\frac{\partial f}{\partial y}} dy$$

So, $\Delta f \approx (2x+3y)\Delta x + (3x-2y)\Delta y$

so e.g. what's $f(2.1, -1.1) - f(2, -1)$?

$x = 2 \quad \Delta x = 0.1$

$y = -1 \quad \Delta y = -0.1$

$= f(x+\Delta x, y+\Delta y) - f(x,y)$

$= \Delta f$

$$\Delta f \approx (2\cdot 2 + 3\cdot(-1))\cdot(0.1) + (3\cdot 2 + (-2)(-1))(-0.1)$$

$$= 1\cdot 0.1 + 8\cdot(-0.1) = -0.7$$

Remark The approximation $f(x,y) \approx L(x,y)$ is valid near (x_0, y_0) only if $f(x,y)$ is differentiable.

How to detect when a function is differentiable?

It's sufficient to check that f_x and f_y both exist and are continuous at (x_0, y_0).