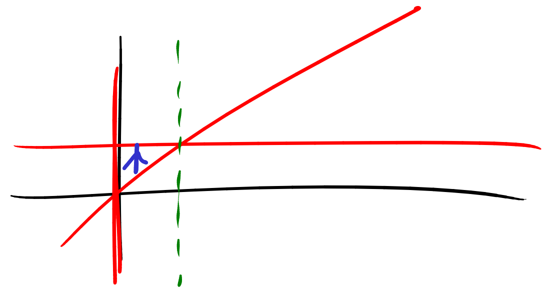
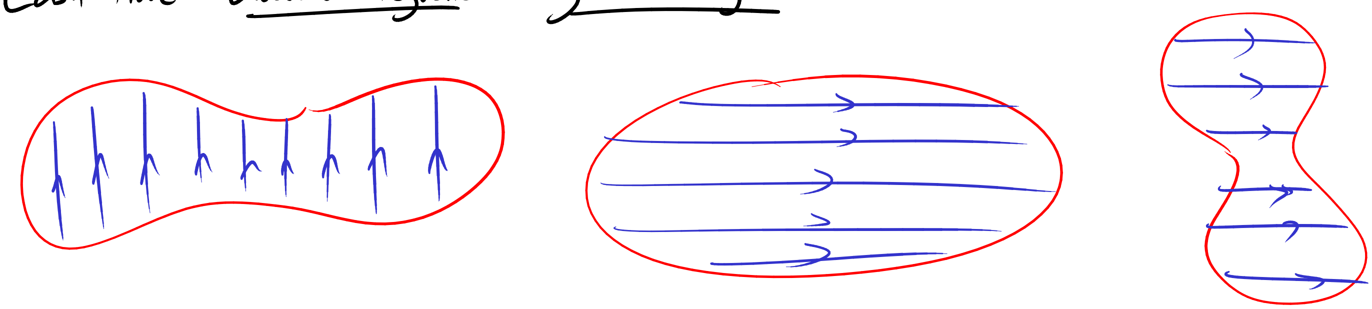
Lecture 23

20 Nov 2010

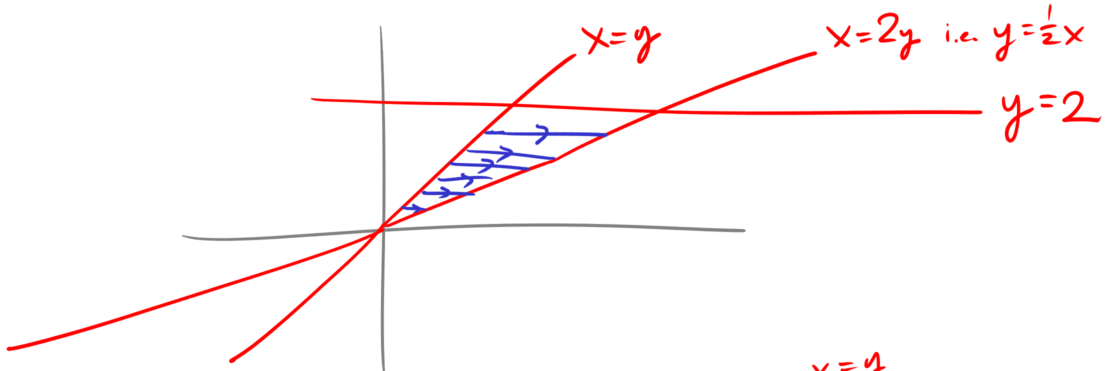
$$\int_0^1 \int_1^x \cdots \; dy\, dx$$



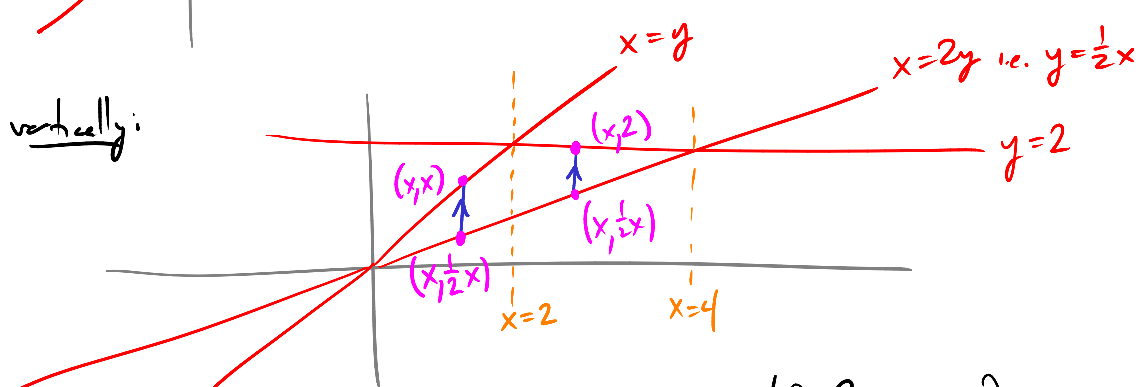
Last time: double integrals over general regions



Ex $$\int_0^2 \int_y^{2y} xy \, dx\, dy$$ — how to rewrite this integral in the opposite order?



To slice vertically:



$$\int_0^2 \left[\int_{\frac{1}{2}x}^{x} xy \, dy \right] dx + \int_2^4 \left[\int_{\frac{1}{2}x}^{2} xy \, dy \right] dx$$

Both methods give the same answer.

Integration in Polar Coordinates

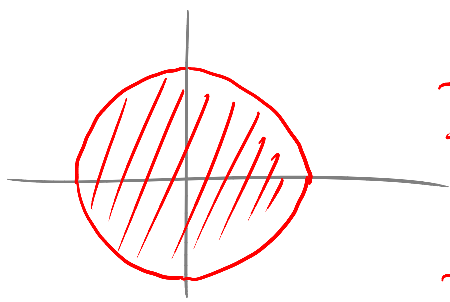
$x = r\cos\theta \qquad r^2 = x^2+y^2$

$y = r\sin\theta \qquad \tan\theta = y/x$

When should we use (r,θ) in the $\int$ instead of (x,y)?

Rough answer: when D has "circular" shape

Ex

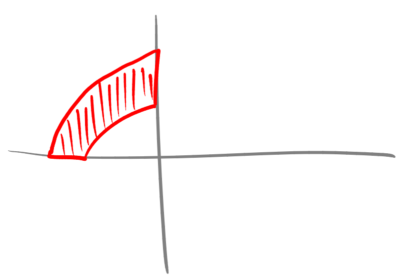


$D = \{(x,y) : x^2+y^2 \leq 4\}$

can also be described as

$D = \{(r,\theta) : r \leq 2\}$

Ex



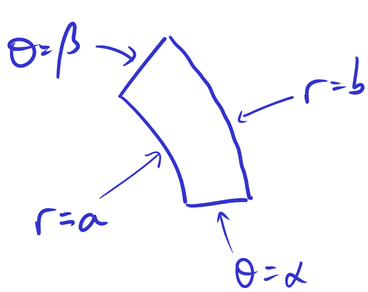
$D = \{1 \leq x^2+y^2 \leq 4, \ x \leq 0, \ y \geq 0\}$

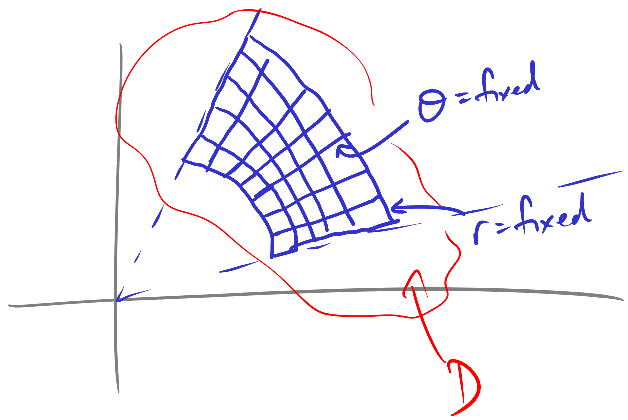
or

$D = \{(r,\theta) : 1 \leq r \leq 2, \ \frac{\pi}{2} \leq \theta \leq \pi\}$

To do an $\int$ in polar coords:

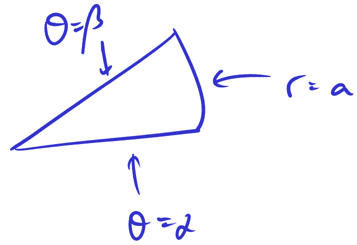
Break D into small pieces looking like





What is the area of each piece?

A wedge



has area

$$(\pi a^2)\left(\frac{\beta-\alpha}{2\pi}\right) = \frac{1}{2}a^2(\beta-\alpha)$$

so our piece has area $\frac{1}{2}b^2(\beta-\alpha) - \frac{1}{2}a^2(\beta-\alpha)$

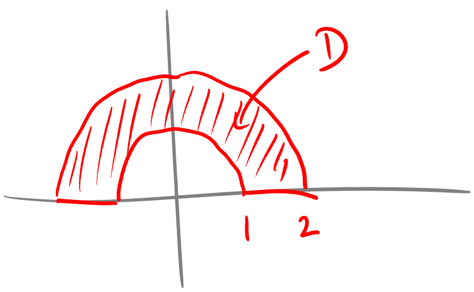
$$= \frac{1}{2}(b^2-a^2)(\beta-\alpha) = \frac{1}{2}(b-a)(b+a)(\beta-\alpha)$$

$$\approx \frac{1}{2}(dr)(2r)(d\theta)$$

$$= r\,dr\,d\theta$$

remember this r!

Ex



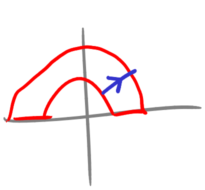
$$\iint_D (3x+4y^2)\,dA$$

$x = r\cos\theta$
$y = r\sin\theta$

$$D = \{(\theta, r) : 1 \le r \le 2,\ 0 \le \theta \le \pi\}$$

so we have

$$\int_{\theta=0}^{\theta=\pi}\int_{r=1}^{r=2} (3r\cos\theta + 4r^2\sin^2\theta)\,r\,dr\,d\theta$$



$$= \int_{\theta=0}^{\theta=\pi}\int_{r=1}^{r=2} 3r^2\cos\theta + 4r^3\sin^2\theta\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=\pi} \Big[r^3\cos\theta + r^4\sin^2\theta\Big]_{r=1}^{r=2}\,d\theta$$

$$= \int_{\theta=0}^{\theta=\pi} 7\cos\theta + 15\sin^2\theta\,d\theta$$

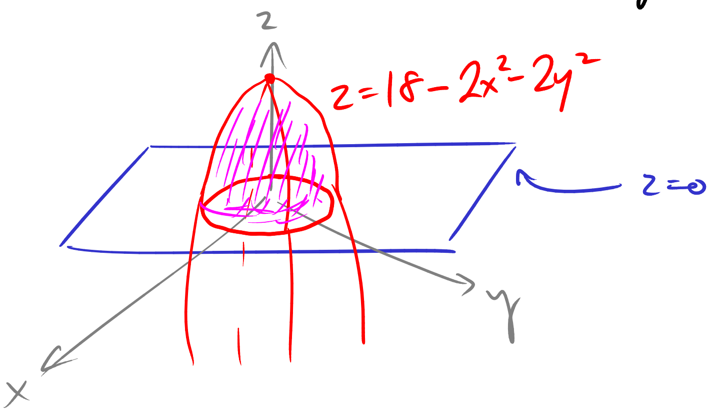
$$= \int_{\theta=0}^{\theta=\pi} 7\cos\theta + 15\left(\tfrac{1}{2}(1-\cos 2\theta)\right) d\theta$$

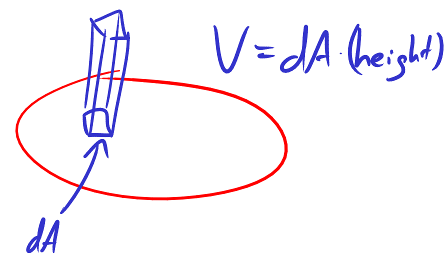
$$= \int_{\theta=0}^{\theta=\pi} 7\cos\theta + \frac{15}{2} - \frac{15}{2}\cos 2\theta \; d\theta$$

$$= \left. 7\sin\theta + \frac{15}{2}\theta - \frac{15}{4}\sin 2\theta \right]_{\theta=0}^{\theta=\pi}$$

$$= \frac{15}{2}\pi$$

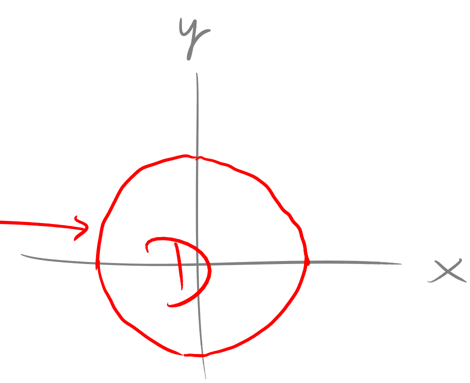
Ex Find the volume of the region bounded by $z = 18 - 2x^2 - 2y^2$, $z = 0$





This region lies over a domain in the x-y plane:

$0 = 18 - 2x^2 - 2y^2$



The volume we want is

$$\iint_D (\text{height})\, dA = \iint_D 18 - 2x^2 - 2y^2 \, dA$$

boundary of D is $18 - 2x^2 - 2y^2 = 0$ ie $x^2 + y^2 = 9$

ie $r = 3$

so $D = \{(r,\theta) : 0 \le r \le 3\}$

so the volume is $\int_0^{2\pi}\int_0^3 (18-2x^2-2y^2)\,r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^3 \left(18-2(r\cos\theta)^2-2(r\sin\theta)^2\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 (18-2r^2\cos^2\theta-2r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 (18-2r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 18r-2r^3\,dr\,d\theta$$

$$= \int_0^{2\pi} 9r^2-\frac{1}{2}r^4\Big]_{r=0}^{r=3}\,d\theta$$

$$= \int_0^{2\pi}\left(81-\frac{81}{2}\right) d\theta$$

$$= \int_0^{2\pi}\frac{81}{2}\,d\theta$$

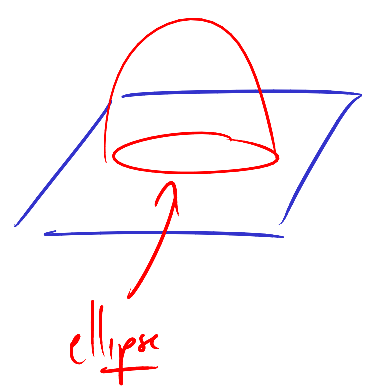
$$= \frac{81}{2}\theta\Big]_0^{2\pi} = \underline{\underline{81\pi}}$$

What if instead we had $z = 18-x^2-4y^2$?

Polar coords won't help.

But, could use elliptic coordinates

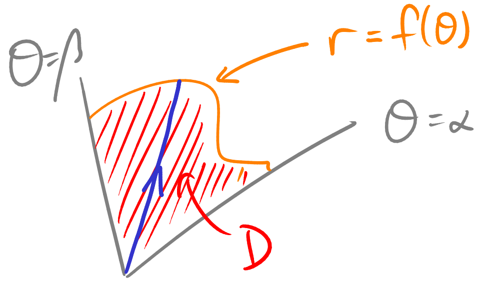
$$x = 2r\cos\theta$$
$$y = r\sin\theta$$



Then our equation becomes $z = 18 - 4r^2\cos^2\theta - 4r^2\sin^2\theta$
$$= 18 - 4r^2$$

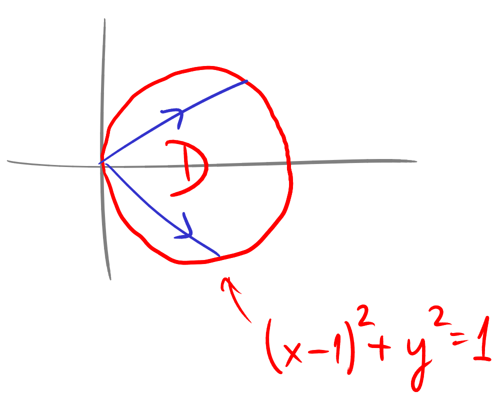
But, what to use for dA in elliptic coordinates? See next lecture!

Ex Find area "under" a curve given in polar coords



$$A = \iint_D 1 \cdot dA = \int_\alpha^\beta \left[\int_0^{f(\theta)} 1 \cdot r\,dr\right] d\theta$$
$$= \int_\alpha^\beta \left[\left[\frac{1}{2}r^2\right]_0^{f(\theta)}\right] d\theta$$
$$= \int_\alpha^\beta \frac{1}{2} f(\theta)^2\, d\theta$$
(we saw this formula before!)

Ex



$f(x,y) = r^2$

We want the integral $\iint_D r^2\, dA$

(moment of inertia around the origin)

Try using polar coordinates: $(x-1)^2+y^2=1$
$$(r\cos\theta-1)^2 + (r\sin\theta)^2 = 1$$

$$r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\sin^2\theta = 1$$
$$r^2 - 2r\cos\theta = 0$$
$$r = 2\cos\theta$$

So $D = \{0 \le r \le 2\cos\theta,\ -\pi/2 \le \theta \le \pi/2\}$

Our int. is $\displaystyle\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2 \cdot r\, dr\, d\theta$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^3\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{4}r^4 \Big]_{r=0}^{r=2\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{4}(2\cos\theta)^4\, d\theta$$

$$= 4\int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta$$

$$= 4\int_{-\pi/2}^{\pi/2} \cos^2\theta \cdot \cos^2\theta\, d\theta$$

$$= 4\int_{-\pi/2}^{\pi/2} \frac{1}{2}(1+\cos 2\theta) \cdot \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 1 + 2\cos 2\theta + \cos^2 2\theta\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 1 + 2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta)\, d\theta$$

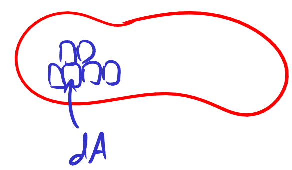
$$= \int_{-\pi/2}^{\pi/2} \frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta \, d\theta$$

$$= \ldots = \frac{3}{2}\pi$$

[average of $\cos 2\theta$, $\cos 4\theta$ as θ varies by π are both zero]

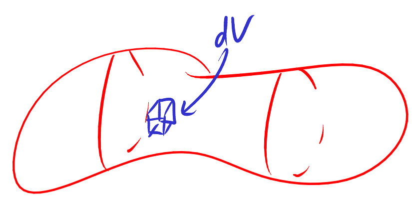
We spent a while on double integrals

$$\iint_D f(x,y)\, dA$$



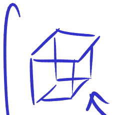
Very similarly could consider triple integrals

$$\iiint_D f(x,y,z)\, dV$$



e.g. if we have a fluid whose density at (x,y,z) is $\rho(x,y,z)$

the total mass of the fluid in the domain D is $\iiint_D \rho(x,y,z)\, dV$

[ mass in here: $\rho(x,y,z)\, dV$]

To compute it: use $dV = dx\, dy\, dz$ in Cartesian coords

but can also use spherical polar, cylindrical, ...

Next time: How to determine dV in any coordinates!