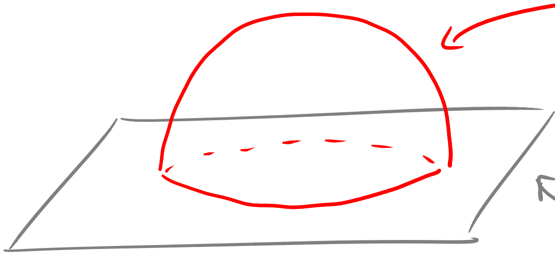
# Lecture 24

25 Nov 2014

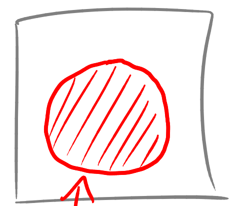
Q: how to do the HW pb involving volume of a sphere with core drilled out?

$x^2+y^2+z^2=16$

$z=\sqrt{16-x^2-y^2}$



$z=0$



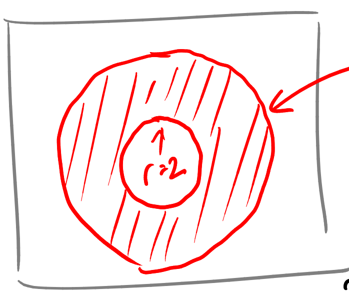
$x^2+y^2=16$

i.e. $r=4$

Volume of top ½ of sphere would be $\displaystyle\iint_D \sqrt{16-x^2-y^2}\,dA$

For sphere with hole cut out,

similar




$r=4$

$$D=\left\{\begin{array}{l}2\le r\le 4,\\ 0\le\theta\le 2\pi\end{array}\right\}$$

so volume of top half $=\displaystyle\int_{\theta=0}^{2\pi}\int_{r=2}^{4}\sqrt{16-r^2}\;r\,dr\,d\theta$

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Administrative remarks:

No discussion sessions tomorrow
  office hours

Last HW will be due next Thu morning

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# Change of Coordinates (Ch 15.10)

We've been doing double $\int$'s in various coordinate systems, e.g.

$(x, y)$ $\qquad dA = dx\,dy$

$(r, \theta)$ $\qquad x = r\cos\theta$, $y = r\sin\theta$ $\qquad dA = r\,dr\,d\theta$

How about other coordinates, e.g. elliptic $\qquad x = 3r\cos\theta$, $y = 4r\sin\theta$

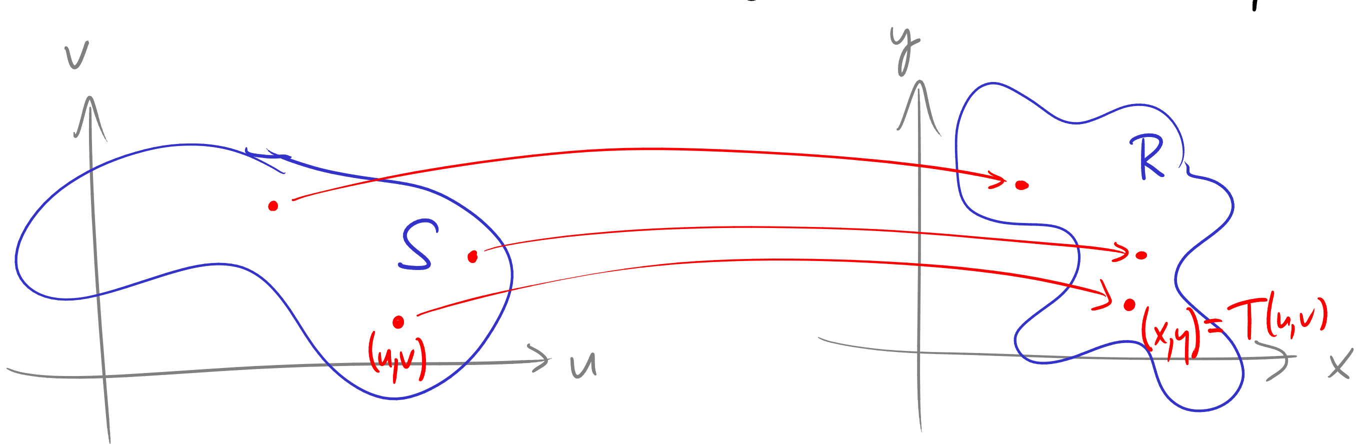
parabolic $\qquad x = \sigma\tau$, $y = \frac{1}{2}(\tau^2 - \sigma^2)$

$\vdots$

All these are examples of coordinate transformations

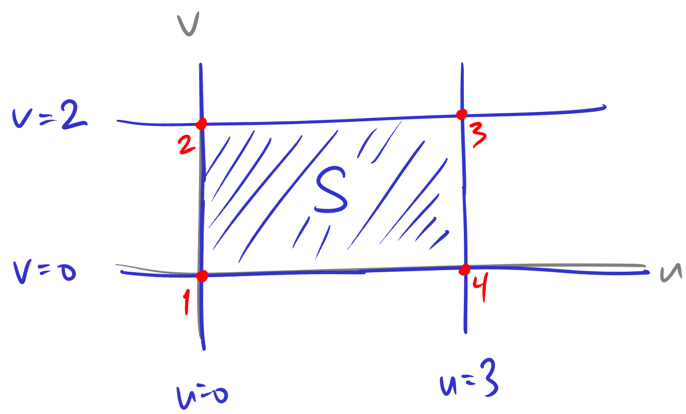
$$\left.\begin{aligned} x &= g(u,v) \\ y &= h(u,v) \end{aligned}\right\} \text{ write this as } (x,y) = T(u,v) \qquad T = (g,h)$$

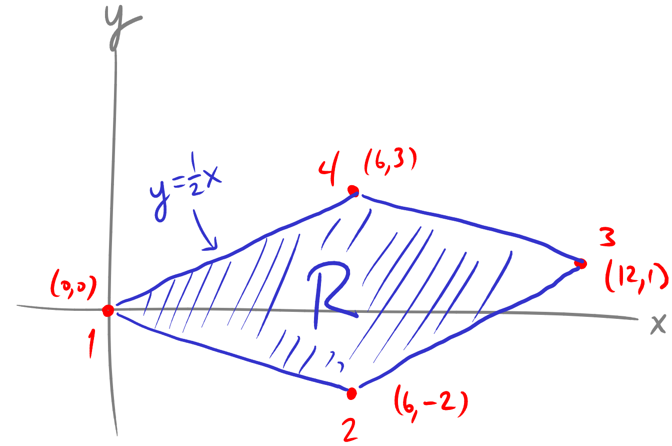
T is a function whose domain and range are both subsets of the plane.



Ex Say $S = \{0 \le u \le 3,\ 0 \le v \le 2\}$

and $T$ is the transformation $x = 2u + 3v$, $y = u - v$




[e.g. $v = 0$ in $x$-$y$ coordinates:

$x = 2u$, $y = u$ i.e. $x = 2y$ or $y = \frac{1}{2}x$]

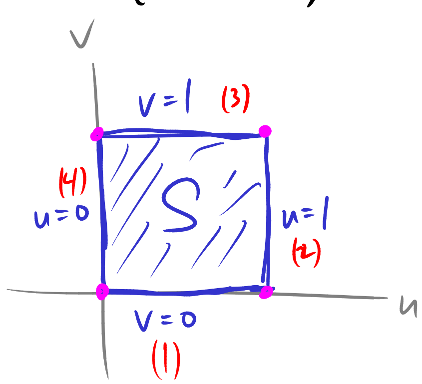
$R = T(S)$

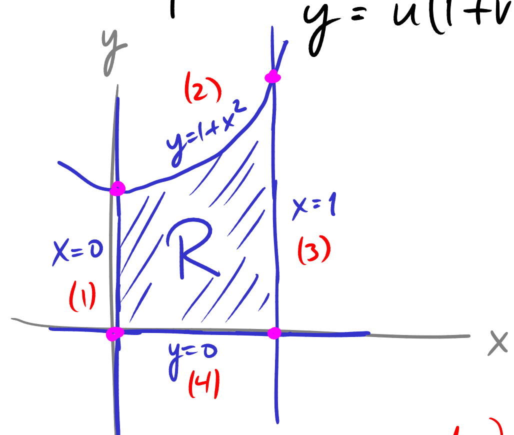
Remark: If $T$ is linear (only involves $1^{st}$ powers of $y$ and $x$) then it transforms straight lines to straight lines.

$\Rightarrow$ transforms quadrilaterals to other quadrilaterals.

But if $T$ is not linear it may act in a more complicated way...

Ex $S = \{0 \le u \le 1,\ 0 \le v \le 1\}$  $T$: $x = v$, $y = u(1 + v^2)$




(1) $\left.\begin{array}{l} x = 0 \\ y = u \end{array}\right\} \to x = 0$

(2) $\left.\begin{array}{l} x = v \\ y = 1 + v^2 \end{array}\right\} \to y = 1 + x^2$

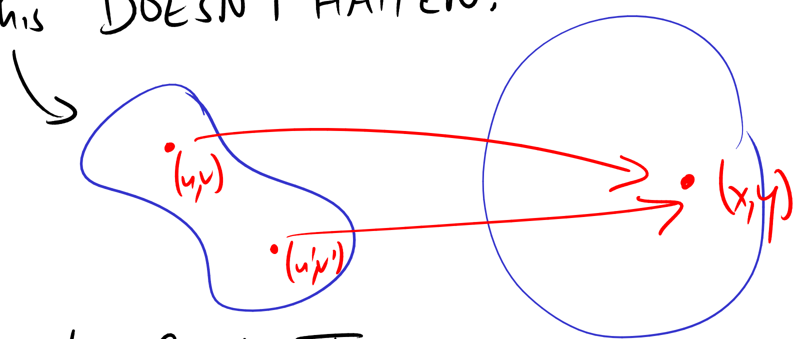
(3) $\left.\begin{array}{l} x = 1 \\ y = 2u \end{array}\right\} \to x = 1$

(4) $\left.\begin{array}{l} x = v \\ y = 0 \end{array}\right\} \to y = 0$

So, coordinate xform can change the shape of the domain — make it more complicated or simpler.

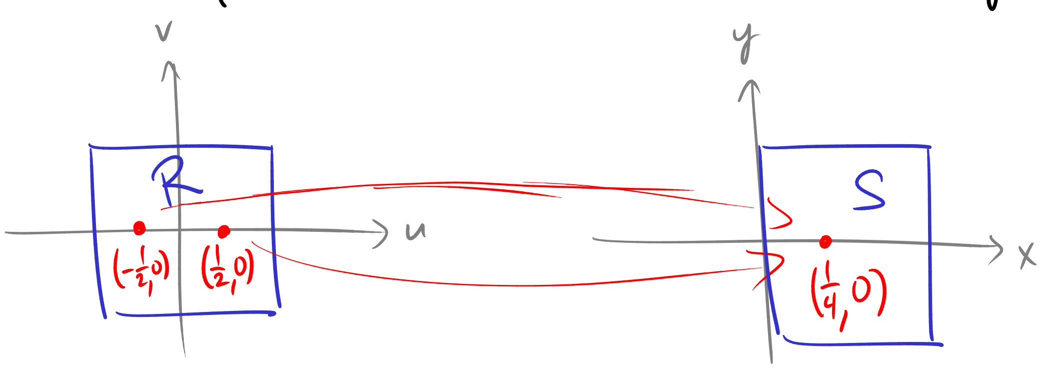
**Remark** We say the transformation $T$ is "1-1" if it never takes 2 different points $(u,v)$ and $(u',v')$ to the same point $(x,y)$.

i.e. this DOESN'T HAPPEN:




An example of a transformation $T$ which is not 1-1:

$R = \{ |u| \le 1, |v| \le 1 \}$ $\qquad T: \begin{array}{l} x = u^2 \\ y = v \end{array}$




$S: \{ 0 \le x \le 1, |y| \le 1 \}$

If $T$ is 1-1 then we can invert it:

$T$ gives $x$ and $y$ as functions of $(u,v)$

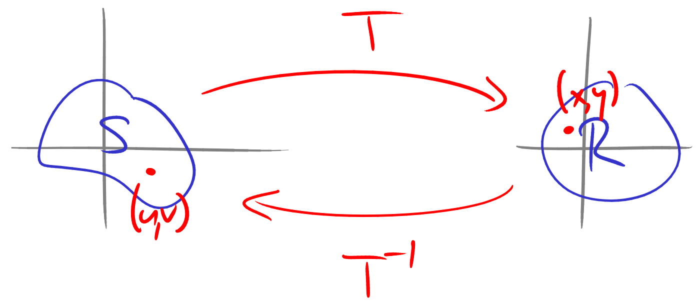
can solve for $u,v$ as functions of $(x,y)$

get a new transformation $T^{-1}$

$$T(u,v) = (x,y)$$

$$T^{-1}(x,y) = (u,v)$$

$$T^{-1}(T(u,v)) = (u,v)$$




## Change of variables in a double integral:

$T$ given by $\left.\begin{aligned} x &= g(u,v) \\ y &= h(u,v) \end{aligned}\right\} \longrightarrow$ matrix of partial derivatives

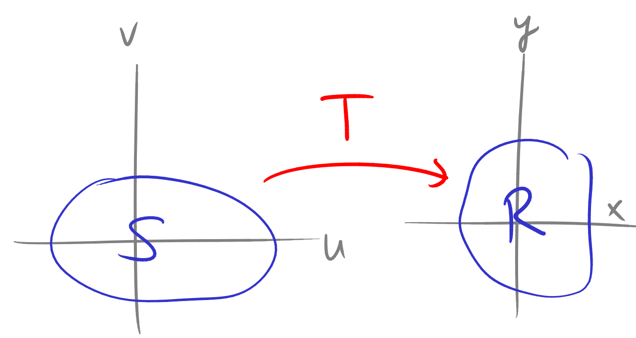
$$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} g_u & h_u \\ g_v & h_v \end{bmatrix}$$

Define the Jacobian of $T$ to be the determinant of this matrix:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

**Fact** If $R = T(S)$ and $T$ is 1-1 giving $\begin{aligned} x &= x(u,v) \\ y &= y(u,v) \end{aligned}$

then




$$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

i.e. "$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$"

**Ex** When $T$ is $\begin{aligned} x &= v\cos u \\ y &= v \sin u \end{aligned}$

$(u,v) = (\theta, r)$

$r > 0$

then $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -v \sin u & v \cos u \\ \cos u & \sin u \end{vmatrix}$$

$$= (-v \sin u)(\sin u) - (v \cos u)(\cos u)$$

$$= -v \sin^2 u - v \cos^2 u$$

$$= -v \qquad \text{and } v > 0$$

so $$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = v$$

Our formula says $$dx\, dy = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

ie $$dx\, dy = v\, du\, dv \qquad (u,v) = (\theta, r)$$

ie $$dx\, dy = r\, d\theta\, dr$$

the area element in polar coordinates ✓

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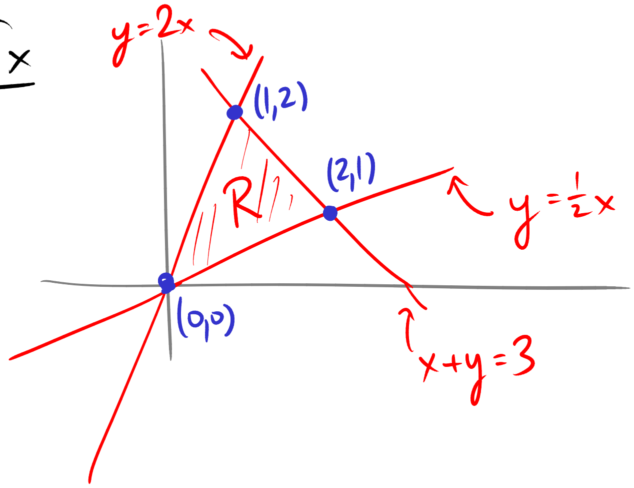
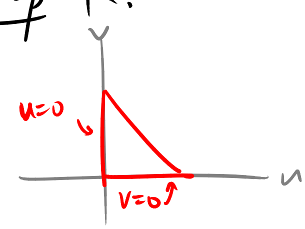
<u>Ex</u>




What is $\iint_R x\, dA$ ?

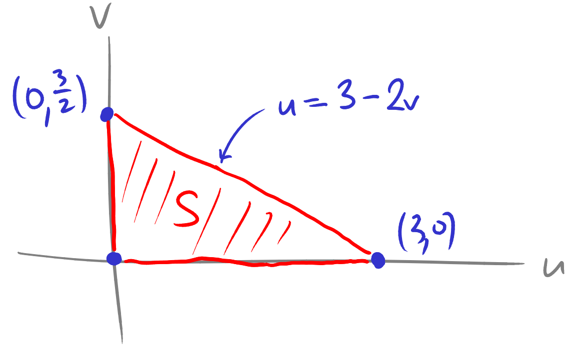
To do this by horiz/vert slices, we'd have to <u>split up</u> R.

Let's find new coords $(u,v)$ where the domain looks like



So: try $\quad u = 2x - y$

$\qquad\qquad v = y - \frac{1}{2}x$




$$\iint_R x \, dx\, dy = \iint_S x \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

Need to know $x, y$ in terms of $u, v$. Solve $\begin{aligned} u &= 2x - y \\ v &= y - \tfrac{1}{2}x \end{aligned}$ for $x, y$:

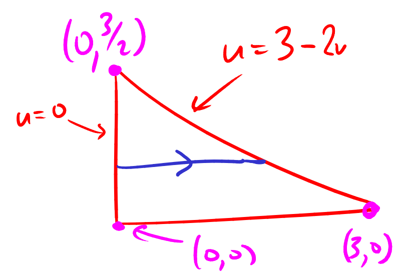
...

get $\quad x = \frac{2}{3}(u+v)$

$\qquad y = \frac{1}{3}(u + 4v)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{4}{3} \end{vmatrix} = \frac{2}{3}\cdot\frac{4}{3} - \frac{2}{3}\cdot\frac{1}{3} = \frac{2}{3}$$

Our integral is
$$\int_{v=0}^{v=3/2} \int_{u=0}^{u=2v-3} \frac{2}{3}(u+v) \cdot \frac{2}{3} \, du\, dv$$



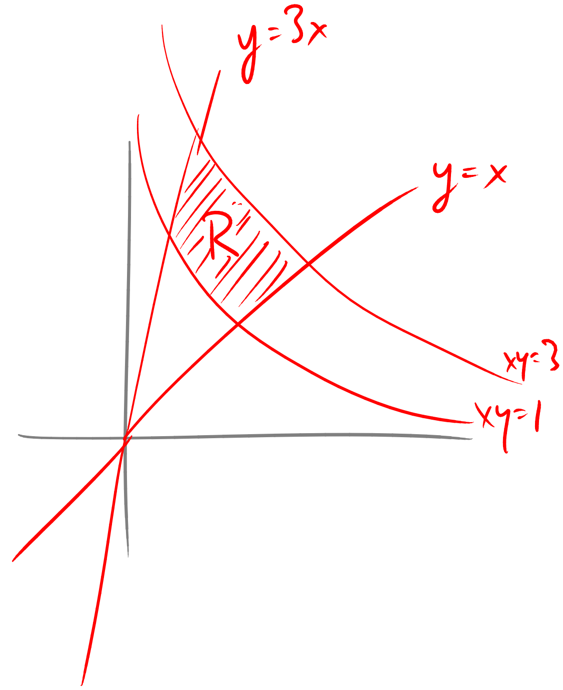

$$= \cdots = \frac{3}{2}$$

Ex $\displaystyle\iint_R xy \, dA \qquad$ R bounded by $\quad y = x$

$y = 3x$

$xy = 1$

$xy = 3$

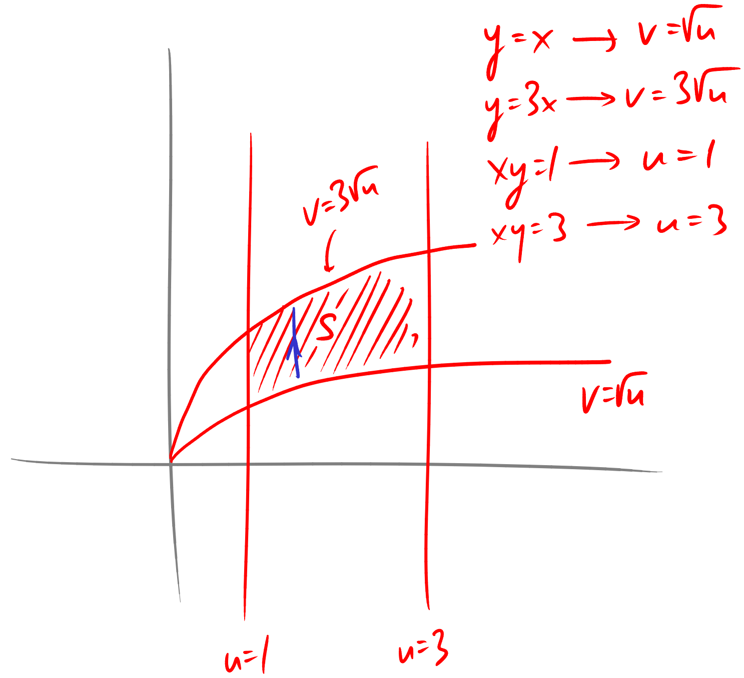



Coord change:

$u = xy$

$v = y$

it's 1-1: $y = v, \quad x = \frac{u}{v}$




$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$

So $$\iint_R xy \, dx\,dy = \iint_S \left(\frac{u}{v}\right)(v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \iint_S u \cdot \frac{1}{v} \, du\,dv$$

$$= \int_1^3 \int_{\sqrt{u}}^{3\sqrt{u}} \frac{u}{v} \, du\,dv$$

$$= \cdots = \underline{4 \ln 3}$$