RESEARCH STATEMENT

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1. Background and motivation

I work in differential topology, dynamics, and geometric group theory. More precisely, I study discrete subgroups of Lie groups via studying geometric structures on manifolds. This approach has its roots with Klein's Erlangen program, which first formally proposed the idea of studying geometries by making use of their groups of symmetries. In this language, a *geometry* consists of a model space together with a transitive group action, where the meaningful geometric notions are those preserved by the group action. A manifold carries a geometric structure if its charts lie in the model space, with transition maps coming from the group action. For example, each surface can be endowed with exactly one of three homogeneous Riemannian geometries given by the uniformization theorem – it is $\mathbb{E}^2$ if the surface has 0 Euler characteristic, $\mathbb{S}^2$ if the Euler characteristic is positive, and $\mathbb{H}^2$ if it is negative. These surfaces can be realized as quotients of the model spaces by an action of a discrete group in their group of symmetries. One of the achievements of the field is Perelman's proof of Thurston's geometrization conjecture about classifying the geometric structures supported by three-manifolds.

In the modern study of geometric structures on manifolds, and in my own research, we commonly use tools such as Lie theory, projective and (pseudo-)Riemannian geometry, Higgs bundles, algebraic geometry, and algebraic topology. The framework of higher-dimensional geometries provides a new perspective on objects such as Hitchin representations and higher Teichmuller spaces, using tools such as Anosov representations [Lab06], [GW12] and positive and maximal representations, to better understand them.

My work mostly concerns *affine geometry,* a non-Riemannian geometry modeled on real affine n-space $\mathbb{A}^n$ with the structure group of affine transformations $\mathrm{Aff}(n) = \mathrm{GL}_n \mathbb{R} \ltimes \mathbb{R}^n$. My main contribution to date is in constructing new examples and deepening the understanding of proper affine manifolds. I introduce a construction called *higher strip deformations* [ŽK22] to provide new examples of free groups acting on $\mathbb{A}^n$.

One of the motivating open problems in affine geometry is the Auslander conjecture [Aus64], an affine analogue of Bieberbach's celebrated theorem stating that all complete Euclidean manifolds have virtually abelian fundamental groups. It proposes that if $\Gamma < \mathrm{Aff}(n)$ is a discrete group acting on $\mathbb{A}^n$ properly discontinuously and cocompactly, then Γ is virtually solvable. The Auslander conjecture is known to be true for $n \leq 6$, [FG83], [AMS12]. Some observations about affine actions of free groups on $\mathbb{A}^7$ in my work could in the future relate to the Auslander conjecture in dimension 7, the next open case.

Margulis [Mar84], [Mar83] constructed the first examples of finitely generated free groups Γ in $\mathrm{Aff}(3)$ acting properly discontinuously on $\mathbb{A}^3$. The quotients $\Gamma\backslash \mathbb{A}^3$, called Margulis space-times, are non-compact complete affine manifolds with non-virtually solvable fundamental groups, answering Milnor's [Mil77] question about the necessity of the cocompactness condition in the Auslander conjecture. Drumm and Drumm-Goldman [Dru92], [DG99] showed that in fact there are many Margulis space-times: any discrete free group in $\mathrm{SO}(2,1)$ can be the linear part of a proper affine action. Making use of the interplay between hyperbolic geometry and affine geometry in three dimensions, Danciger, Gueritaud, and Kassel [DGK16b] took this further and, for a discrete free group $\Gamma < \mathrm{SO}(2,1)$, used strip deformations of the hyperbolic surface $\Gamma\backslash\mathbb{H}^2$ to describe the cone of proper affine actions. It is their approach that I generalize in my work.

In higher dimensions, the known picture of the theory of affine actions is much more mysterious. Abels, Margulis, and Soifer [AMS02] showed that in some sense, $\mathrm{SO}(2n, 2n-1)$ is a natural group to find linear parts in. Smilga [Smi14] constructed fundamental domains in $\mathbb{A}^{2n,2n-1}$ for a family of proper affine deformations of certain free groups in $\mathrm{SO}(2n, 2n-1)$. Using different methods, my work elucidates a subtlety of Smilga's construction not present in dimension 3.

2. Past results

In my work so far I have constructed new examples of proper affine deformations of Fuchsian free groups in $\mathrm{SO}(2n, 2n-1)$ for any n via a construction called a *higher strip deformation*, closely related to infinitesimal strips from [DGK16b]. My main result is a constructive proof of the following:

Theorem 1 ([ŽK22]). *Let $S = \Gamma\backslash\mathbb{H}^2$ be a noncompact convex cocompact surface and $\sigma_{4n-1} : \mathrm{PSL}_2\,\mathbb{R} \to \mathrm{SO}(2n, 2n-1)$ the irreducible representation. Then $\sigma_{4n-1}(\Gamma)$ admits an open cone of cocycles determining proper affine actions on $\mathbb{A}^{2n,2n-1}$.*

The proper actions in Theorem 1 are constructed directly from geometric data on the surface S. The proof gives a concrete family of deformations to work with.

A *strip system* on S is the data of a collection of properly embedded arcs $\underline{a}$ on S, a point $p_i \in a_i$ for each arc $a_i \in \underline{a}$ called the *waist,* angles $\theta_i \in [-\pi, \pi]$, and real numbers w_i. We turn this data into an affine deformation of $\sigma_{4n-1}(\Gamma)$ by explicitly describing the translation part of each $\gamma \in \Gamma$.

It is enough to describe a higher strip deformation along one arc a with waist p and angle θ, as a general higher strip deformation is a linear combination of these. Let $\tilde{a}$ be the lift of a to $\tilde{S} \cong \mathbb{H}^2$, and choose a basepoint $x_0 \in \mathbb{H}^2 \setminus \tilde{a}$. For each arc $\tilde{a}_j \in \tilde{a}$, let $\eta_j \in \mathrm{PSL}_2\,\mathbb{R}$ be the unit-speed translation whose invariant axis crosses $\tilde{a}_j$ at the lift $\tilde{p}_j \in \tilde{a}_j$ of p and makes the angle θ with the perpendicular to $\tilde{a}_j$. Let c be a path between x_0 and $\gamma \cdot x_0$. The translation $u(\gamma)$ we associate to $\sigma_{4n-1}(\gamma)$ is the sum of (appropriately normed and oriented) eigenvalue 1 eigenvectors of $\sigma_{4n-1}(\eta_j)$ associated to each $\tilde{a}_j$ that c crosses. This is illustrated in Figure 1.

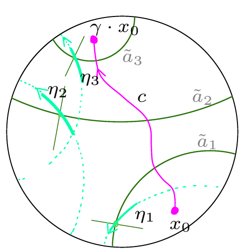

Figure 1. An illustration of a lift of an arc that the curve c crosses three times. In this case, the translation $u(\gamma)$ associated to γ would be $x^0(\sigma_{4n-1}(\eta_1)) + x^0(\sigma_{4n-1}(\eta_2)) + x^0(\sigma_{4n-1}(\eta_3))$, where $x^0(A)$ denotes the appropriately scaled and oriented eigenvalue 1 eigenvector of A, also called the neutral vector. The construction depends on the choice of basepoint x_0 up to conjugation, and does not depend on the choice of path c.

In the case of $n = 1$, we recover infinitesimal strip deformations from [DGK16b] by choosing all angles θ_i to be 0, and infinitesimal earthquakes by choosing the angles θ_i to be $\pm\frac{\pi}{2}$.

The tool for studying affine deformations of subgroups Γ in $\mathrm{SO}(2n, 2n-1)$ is *the Margulis invariant*. Given a Γ-cocycle $u : \Gamma \to \mathbb{R}^{2n,2n-1}$, it is a class function $\alpha_u : \Gamma \to \mathbb{R}$, recording the signed translation distance along an invariant axis of each $\gamma \in \Gamma$. The opposite-sign lemma in [Mar84], [Mar83] states that as soon as there exist elements γ, $\eta \in \Gamma$ so that $\alpha_u(\gamma)\alpha_u(\eta) \leq 0$, the affine action determined by u is not proper. The invariant was later extended to a diffuse version defined on geodesic currents by Labourie [Lab01], and used in [GLM04] by Goldman, Labourie, and Margulis to show that an affine action with Fuchsian linear part is proper exactly when the diffuse Margulis invariant only takes positive (or only negative) values. These results were generalized to the case of Anosov linear part by Ghosh and Treib in [GT22].

It is possible to directly compute the Margulis invariant of a higher strip deformation by summing up contributions to $\alpha_u(\gamma)$ corresponding to each intersection between $\underline{a}$ and (the geodesic curve representing) γ. This contribution depends only on the relative positions of the axis of γ and the axis of η_j, and can be expressed as a rational function in terms of the cross-ratio t of their endpoints. The rational function in question is a hypergeometric function with one simple pole and $2n-1$ simple zeros. A sketch of the behavior of the function is described in Figure 2.

For $n = 1$, any choice of waists for a filling arc system with choice of angles $\theta_i = 0$ determines a proper affine deformation, as the contribution upon *each* crossing to the Margulis invariant in that case is positive for *all* possible arrangements of η and γ. However, for larger n, the situation is more delicate. Even if we choose the angles θ_i to all be 0, we can achieve relative positions of η and γ with both positive and negative contributions to the Margulis invariant – there are in total $2n$ relative positions where the sign of the Margulis invariant flips. In the proof of Theorem 1, I show that a choice of waist outside the convex core of S and a careful choice of angles θ_i still gives an open family of proper affine deformations. These carefully constructed strip systems produce proper affine deformations for each n.

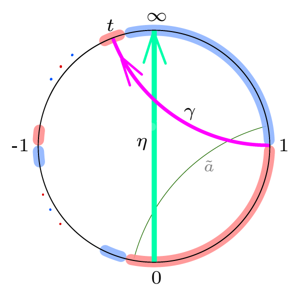

FIGURE 2. The Margulis invariant contribution upon one crossing is $\frac{1}{(t-1)^{2n-1}} \sum_{j=0}^{2n-1} \binom{j}{2n-1}^2 t^j$. This rational function extends to $\mathbb{RP}^1$, has $2n-1$ negative simple real zeros, and a pole at 1. As n gets bigger, the largest modulus of a zero approaches infinity. The picture on the left demonstrates regions of positivity (blue) and negativity (red) of the contribution function if we fix η (green) and the backwards endpoint of γ (pink).

Choosing appropriate arcs and translations η recovers Smilga's construction from [Smi14] in the Fuchsian case. The waists of such a strip system land inside the convex core of S, and are therefore harder to analyze due to the sign-switching behavior of the Margulis invariant. Given a dynamical contracting condition on the linear part not needed in the case of $n = 1$, Smilga constructs fundamental domains for such actions. I construct concrete examples of affine deformations of Smilga type where the dynamics condition is not satisfied and whose Margulis invariants take both positive and negative values, showing the necessity of restricting the linear part of deformations of Smilga type. I write down a thinness condition on the geometry of the convex core of S that guarantees properness of an action even when the waists are inside the convex core, providing an alternate perspective on Smilga's contracting assumption. The Margulis invariant perspective also explains why the dynamics condition is only needed for $n > 1$.

Note that by [Mes07], [DZ19, Lab22], Fuchsian surface groups cannot have proper affine deformations. However, virtually free groups can. By considering strip systems that are equivariant with respect to a finite group action, it is possible to construct higher strip deformations of virtually free groups, such as $\mathrm{PSL}_2\,\mathbb{Z} \cong \mathbb{Z}/2 \star \mathbb{Z}/3$. For $\mathrm{PSL}_2\,\mathbb{Z}$ there is only one strip system, which for $n = 1, 2$ realizes the (up to scaling and coboundaries) unique cocycle. The Margulis invariant can be defined for parabolic elements, but the [GLM04] result only applies to groups with fully loxodromic linear part and $\mathrm{PSL}_2\,\mathbb{Z}$ contains parabolic elements; we cannot currently make concrete conclusions about the properness of the action determined by this cocycle. However, all the Margulis invariants of the loxodromic elements are uniformly positive, leading us to conjecture that all nontrivial affine deformations of $\sigma_3(\mathrm{PSL}_2\,\mathbb{Z})$ and $\sigma_7(\mathrm{PSL}_2\,\mathbb{Z})$ are proper. If we open up the cusp of the modular surface $\mathrm{PSL}_2\,\mathbb{Z}\backslash\mathbb{H}_2$, deforming its fundamental group to one with only loxodromic elements, we obtain proper actions of $\mathbb{Z}/2 \star \mathbb{Z}/3$ on $\mathbb{A}^{2,1}$ and $\mathbb{A}^{4,3}$. If we deform even further, making the convex core of $(\mathbb{Z}/2 \star \mathbb{Z}/3)\backslash\mathbb{H}^2$ thin enough, we can construct proper affine actions on any $\mathbb{A}^{2n,2n-1}$. Note that for $n = 1, 2$, all choices of cocycle will in fact give proper affine actions, as the group cohomologies $H^1(\mathrm{PSL}_2\,\mathbb{Z}, \mathbb{R}^3)$ and $H^1(\mathrm{PSL}_2\,\mathbb{Z}, \mathbb{R}^7)$ are both one-dimensional.

3. Work in progress and near-future goals

3.1. **Positive representations.** In joint work with Jean-Philippe Burelle, we construct proper affine deformations of *positive representations* of free groups in $\mathrm{SO}(2n, 2n-1)$, as well as fundamental domains for these actions. We cone off polyhedral fundamental domains for actions of free groups in $\mathrm{SO}(n, n-1)$ on spheres and projective spaces from [BT22]. These can be defined by using a cyclical order on the space of oriented flags $\mathcal{F}^+(\mathbb{R}^{2n,2n-1})$ in $\mathbb{R}^{2n,2n-1}$, giving Schottky domains for positive free groups acting on $\mathbb{S}^{4n-2}$. By appropriately coning them off and translating them in affine space, we obtain *crooked domains* in $\mathbb{A}^{2n,2n-1}$, bounded by crooked hyperplanes, higher-dimensional versions of Drumm's crooked planes. We use them in a similar way to their use in three dimensions as in [Dru92],[DG99], [DGK16b].

If $\mathcal{H} \subset \mathbb{A}^{2n,2n-1}$ is a crooked hyperplane bounding a crooked half-space $\mathcal{H}^+$, we can identify a two-dimensional *stem quadrant* of directions in $\mathcal{H}$ with the property that $\mathcal{H}^+ + v \subseteq \mathcal{H}^+$ for each v in the stem quadrant. Constructing Schottky domains for positive free groups in $\mathrm{SO}(2n, 2n-1)$ gives us instructions for constructing a collection of crooked hyperplanes. In turn, their stem quadrants allow us to identify *allowable affine deformations.* Translated crooked half-spaces determine fundamental domains for an affine action. Observing that an allowable affine deformation has uniformly positive Margulis invariants allows us to conclude that the action is proper on $\mathbb{A}^{2n,2n-1}$, and therefore the crooked domains tile all of $\mathbb{A}^{2n,2n-1}$, yielding a complete affine manifold. This shows

Theorem 2 (in progress)**.** *Let $\Gamma < \mathrm{SO}(2n, 2n-1)$ be a positive free group containing only loxodromic elements. Then the allowable affine deformations of Γ determine proper affine actions on $\mathbb{A}^{2n,2n-1}$, and the crooked half-spaces are fundamental domains for these actions.*

This construction works for any loxodromic positive representation of a free group in $\mathrm{SO}(2n, 2n-1)$, with no further restrictions on the linear part, giving a large family of examples. It also provides fundamental domains for the actions, thus determining the topology of the quotient, making questions about its geometry more tractable.

For the Fuchsian case, we can view affine deformations coming from this stem quadrant construction as limits of higher strip deformations: given an arc a on a surface S, we can use the limit map $\partial\mathbb{H}^2 \to \mathcal{F}^+(\mathbb{R}^{2n,2n-1})$ to construct a crooked hyperplane $\mathcal{H}_a$ in $\mathbb{A}^{2n,2n-1}$. If we let waists on a limit to the boundary, the translational parts associated to a limit to directions in the stem quadrant of $\mathcal{H}_a$. Further, for $n = 1$, infinitesimal strip deformations from [DGK16b] are a special example of these.

3.2. **Actions with parabolics.** As touched upon at the end of Section 2, the Margulis invariant only characterizes properness when the linear part of $\Gamma < \mathrm{SO}(2n, 2n-1) \ltimes \mathbb{R}^{2n,2n-1}$ consists of loxodromic elements. In three dimensions, there are versions of the Margulis invariant for parabolics, see for instance [CD05], and [DGK] describe the cone of all proper actions for a discrete free group in $\mathrm{SO}(2, 1)$, even in the presence of parabolics. My goal is to extend the properness criterion from [GLM04] and [GT22] to discrete subgroups in $\mathrm{SO}(2n, 2n-1)$, potentially containing parabolic elements.

Theorem 3 (in progress)**.** *Let Γ be a discrete subgroup of $\mathrm{PSL}_2\,\mathbb{R}$ and $u : \Gamma \to \mathbb{R}^{2n,2n-1}$ a $\sigma_{4n-1}(\Gamma)$-cocycle. Then the action of (Γ, u) on $\mathbb{A}^{2n,2n-1}$ is proper exactly when the normed Margulis invariant is positive and bounded away from 0 for all loxodromic elements of Γ and takes the value ∞ on parabolic elements.*

One reason that the same methods present in [GLM04] do not work for Fuchsian free groups with parabolics is that the methods in [GLM04] rely on the compactness of the space of geodesic currents on the surface S, which is no longer compact when S is not convex cocompact. In the Fuchsian case of Theorem 3, I circumvent this by taking a more direct and computational approach in approximating a probability measure with closed curves on S, instead of relying on abstract convergence properties.

The situation away from the Fuchsian representations is at the moment less clear. I believe it would be possible to combine two different generalizations of Anosov representation. [GT22] introduce the notion of an *affine Anosov representation*, generalizing the results of [GLM04] to groups with Anosov linear part. In [CZZ22, KL18, Zhu21], the notion of cusped or relative Anosov representations is explored. Marrying these two approaches might lead to a robust notion of cusped Margulis invariants, allowing us to develop an analogue of the [GLM04] properness criterion.

Proving Theorem 3 would allow us to construct new examples of complete affine manifolds, and would let us further explore affine actions of $\mathrm{PSL}_2\,\mathbb{Z}$. Extending it to non-Fuchsian linear parts would also make exploring deformations of positive representations with parabolics from Section 3.1 more tractable. More broadly, many of the results about Margulis space-times in higher dimensions that are currently only known for representations with Anosov linear parts would likely generalize to representations with cusped Anosov linear parts.

4. Further questions and the indeterminate future

4.1. **Shape of the cone of proper actions.** Given a linear part $\Gamma < \mathrm{SO}(2n, 2n-1)$, the Γ-cocycles determining proper affine actions form a convex cone in $H^1(\Gamma, \mathbb{R}^{2n,2n-1})$. For $n = 1$, the cone can be described in terms of the arc complex of the surface $S = \Gamma\backslash\mathbb{H}^2$, as in [CDG09], [DGK16b]. It is also the set of all cocycles u such that the normed Margulis invariant is uniformly positive on all γ represented by *simple* closed curves on S.

In higher dimensions, the picture is more elusive. The simple closed curves no longer suffice, which we can demonstrate with explicit examples of a three-holed sphere group in $\mathrm{SO}(6, 5)$ and a higher strip deformation where the three cuff curves – the only simple closed curves on a three-holed sphere – all have positive Margulis invariants, but some other non-simple curve has a negative Margulis invariant. The greater dependence of the properness of the action on the geometry of the surface we started off with, as in the discussion of deformations of Smilga type in Section 2, further suggests that describing the cone of proper cocycles might be more subtle in higher dimensions, and could, even in the Fuchsian case, depend on more than just the topological type of S. It is nonetheless sensible to ask:

Question 4. Let Γ be a Fuchsian free group in $\mathrm{SO}(2n, 2n-1)$. Is there a natural proper subset of geodesic currents P_n on S such that $\alpha_u(\gamma) > 0$ for each $\gamma \in P_n$ means that u determines a proper affine action on $\mathbb{A}^{2n,2n-1}$?

Question 5. Is there a natural parametrization of the cone of proper affine deformations depending on topological and geometric data on S?

I am running computations in Mathematica in order to find some patterns in small examples, such as Fuchsian three-holed sphere groups acting on $\mathbb{A}^{4,3}$. In the Fuchsian dimension 7 case in particular, there might be a way to leverage the fact that the irreducible representation $\mathrm{PSL}_2\,\mathbb{R} \to \mathrm{PSL}_7\,\mathbb{R}$ factors through the 7-dimensional representation of the exceptional Lie group G_2.

In general, I expect Fock-Goncharov coordinates [FG03], [BD17] could be of use in studying potential parametrizations. Some natural candidates for the set P_n would be curves that lift to simple curves in some finite-sheeted cover, with the number of sheets bounded by a function depending on n.

4.2. **Fundamental domains.** Many methods of constructing complete affine manifolds rely on constructing fundamental domains for group actions on $\mathbb{A}^d$, such as [Smi14], [Dru92], [DG99], [DGK16b], as well as my joint work in progress with Burelle as sketched in Section 3.1. Special cases of higher strip deformations coincide with affine deformations of Smilga type, and therefore his construction of tennis ball fundamental domains – domains obtained by taking conical neighborhoods of disjoint half-planes in the sphere – from [Smi14] works. Limits of higher strip deformations give us the allowable affine deformations from Section 3.1, which also come with fundamental domains.

However, neither the Smilga construction nor the crooked half-spaces construction work for a general proper higher strip deformation; the crooked half-spaces seem to be too rigid, with their stem quadrants too small, and Smilga's tennis ball domains rely too much on the dynamics of the linear part to work for a general higher strip deformation. The combinatorial data of a strip system still appears simple enough that some adaptation of crooked domains seems plausible.

4.3. **Higher strip deformations and affine actions with linear part away from the Fuchsian locus.** In upcoming work with Burelle as discussed is Section 3.1, we can construct proper affine deformations of free groups with any loxodromic positive linear part, and they are in a sense limits of higher strip deformations when the linear part is also Fuchsian. A downside of our construction is that we do not get an *open* cone of proper affine deformations from a choice of arcs. Indeed, for every arc there is only a two-dimensional cone of admissible directions associated to it in every dimension.

By carefully studying the behavior of the limit map of the linear part, it should be possible to extend the definition of a higher strip deformation from the Fuchsian case to the more general Anosov linear part case in a way that gives us an open cone of affine deformations whose properness is easy to check. It is also possible that a good understanding of the setting with more general linear part would rid us of any unnecessary structure muddying the waters in the Fuchsian case, making it easier to answer Questions 4 and 5.

4.4. **Connections to negative curvature.** In three dimensions, infinitesimal strip deformations are derivatives along paths in the Teichmuller space determined by strip deformations on surfaces. Another way to view Margulis space-times is as a rescaled version of AdS_3-manifolds via geometric transitions, making Margulis spacetimes into infinitesimal versions of AdS_3-manifolds, as in [DGK16a]. Using these guidelines, it is natural to ask

Question 6. What is the non-infinitesimal equivalent of higher strip deformations? Is there a natural construction of $\mathbb{H}^{2n,2n-1}$-manifolds corresponding to higher strip deformations that can be described using a strip system? What is the relationship between properness of an affine action and properness of an action on $\mathbb{H}^{2n,2n-1}$?

A direct computation should be feasible, but other tools that might be useful are Fock-Goncharov coordinates on $\mathrm{SO}(2n,2n)$. The hope is that affine deformations in $\mathrm{SO}(2n,2n-1)$ would correspond nicely to some form of bending in $\mathrm{SO}(2n,2n)$ or its symmetric space $\mathbb{H}^{2n,2n-1}$. A concrete connection between higher-dimensional Margulis spacetimes and $\mathbb{H}^{2n,2n-1}$-manifolds would be useful in studying the questions about the cone of proper actions from Section 4.1, as we could study both affine and $\mathbb{H}^{2n,2n-1}$-manifolds.

4.5. **Connections to the Auslander conjecture.** The Auslander conjecture, posing that fundamental groups of closed complete affine manifolds are virtually solvable, is known for manifolds of dimension at most 6. The next natural case to look at is manifolds with linear holonomy in the 7-dimensional representation of G_2. Having a good understanding of different ways for free groups with G_2 linear part to act on $\mathbb{A}^7$ is a first step to untangling the Auslander conjecture in this case. Higher strip deformations, as well as allowable affine deformations from Section 3.1, give a concrete family of examples to work with. The case of higher strip deformations for $\mathrm{PSL}_2\,\mathbb{Z}$ in dimension 7 suggests that there is some special behavior or extra symmetry of strip deformations in dimension 7 not present in higher dimensions which might restrict possible ways for a free group with linear part in $\mathrm{SO}(4,3)$ or G_2 to act properly on affine 7-space, suggesting there is not "enough room" for fitting together enough free groups to make an affine crystallographic group in dimension 7.

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