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Black Holes and Positive Scalar Curvature

Candidate number:
1047693

Supervisor:
Prof. Jason Lotay

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Contents

Introduction

The goal of this paper is to provide a general overview of the known results related to the question: *what shapes can black holes be?* This question was first posed in the context of black holes in four dimensions, and it was answered by Hawking in 1972 with the conclusion that such black holes are topologically spherical. However, since then, has become clear that higher-dimensional black holes need not have spherical topology. A generalisation of Hawking's theorem shows that topologies arising as higher-dimensional black holes need only carry metrics of positive scalar curvature. Thus the majority of this paper investigates topological obstructions to such metrics and outlines their proof methods. We assume knowledge of smooth manifolds, Riemannian geometry, algebraic topology, and the basics of general relativity and representation theory.

1 Black Holes in Higher Dimensions

We begin by examining the first known black hole solution and expanding to higher dimensions. For this section, we use Einstein summation notation.

1.1 The Einstein Equations and the Schwarzschild Solution

Let (M, g) be a Lorentzian manifold modelling a spacetime. We recall the *Einstein field equations* in geometrised units from general relativity:

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2} S g_{\mu\nu} = T_{\mu\nu}$$

where Ric is the Ricci curvature tensor, S is the scalar curvature, and T is the stress-energy tensor of our spacetime. The first nontrivial solution found to them, the Schwarzschild solution, was the 4-dimensional solution:

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Here, $d\Omega^2$ indicates the square of the surface element of the 2-sphere:

$$d\Omega^2 = d\theta^2 + \sin^2\theta \; d\phi^2$$

One question one might ask is whether this solution can be generalised to higher dimensions; that is, whether a metric similar to this one solves Einstein's equations on a manifold of dimension greater than 4 with a Lorentzian signature. In fact, this is possible with only a slight correction. These generalised solutions are known as Schwarzschild-Tangherlini spacetimes, and for an n-dimensional spacetime, they have the form:

$$ds^2 = \left(1 - \frac{2M}{r^{n-3}}\right) dt^2 + \left(1 - \frac{2M}{r^{n-3}}\right)^{-1} dr^2 + r^2 d\Omega_{n-2}^2$$

where $d\Omega_{n-2}^2$ is the surface element of the $(n-2)$-sphere. As one might guess, these spacetimes feature spherical black holes; or at least, the submanifold given by fixing r at the Schwarzschild radius $r = 2M$ is an $(n-2)$-sphere about the origin, and analysis of geodesics in this spacetime show the same behaviour as geodesics in the 4-dimensional Schwarzschild spacetime. More information on these spacetimes can be found in §5.1 of [Hor12]. To discuss the possibility of non-spherical black holes, however, we need a definition of a black hole that does not rely on the Schwarzschild radius or similar coordinate singularity.

1.2 Hawking's Theorem and its Generalisation

The notion of a black hole we will use is called a *marginally outer trapped surface* and is due to Penrose in [Pen65]. We quickly summarise this formalism, as given in §7 of [Hor12].

Let (M^{n+1}, g) be a Lorentzian manifold with n spatial dimensions. Let $V^n \subset M$ be a spacelike hypersurface with induced metric h, and let u be the future directed timelike unit vector field normal to V in M. Finally, let Σ^{n-1} be a compact, bounding hypersurface with induced metric k in V, and let v be the outward-pointing unit normal vector field to Σ in V. Define the following two vector fields in $TM|_\Sigma$:

$$\begin{aligned} l_+ &:= u + v \qquad \text{(points to the future and out)} \\ l_- &:= u - v \qquad \text{(points to the future and in)} \end{aligned}$$

Note that $l_\pm$ are both null. We then define two $(0,2)$-tensors χ_+ and χ_- on Σ as follows:

$$\chi_\pm(X, Y) = g(\nabla_X l_\pm, Y)$$

These tensors are called the *null second forms* of Σ. Their k-traces $\theta_\pm := k^{\mu\nu}(\chi_\pm)_{\mu\nu}$ are called the *null expansion scalars* of Σ. Note that:

$$\theta_\pm = k^{\mu\nu}(\chi_\pm)_{\mu\nu} = \mathrm{div}_\Sigma\, l_\pm$$

and so $\theta_\pm$ measure the divergence of $l_\pm$.

The physical interpretation here is as follows. We think of $l_\pm$ as the initial tangents to light rays emanating from Σ. The expansion scalars, then, measure the deviation of volume of the parallel copy of Σ formed by light from the volume of Σ. One would usually expect $\theta_+ > 0$ and $\theta_- < 0$, as light emitted outwards from a surface defines a larger parallel copy surrounding it, and light emitted inward defines a smaller one. With this formalism, we give our definition for a marginally outer trapped surface, the object we think of as the boundary of a black hole.

Definition 1.2.1. We say that Σ is *outer trapped* if $\theta_+ \leq 0$, and *marginally outer trapped* if $\theta_+ \equiv 0$.

We abbreviate "marginally outer trapped surface" to MOTS. The MOTS is the outermost of all trapped surfaces surrounding the black hole, and corresponds to what physicists call the *apparent horizon*. In stationary spacetimes, this is the same as the event horizon, though they can differ for rotating black holes (the apparent horizon is inside the event horizon in this case; see [HE73]).

In a 4-dimensional spacetime, the presence of a MOTS always indicates the presence of a singularity inside them; hence we are justified in considering them to be black hole boundaries in some sense. This was shown by Penrose in the same paper:

Theorem 1.2.2 (Penrose, [Pen65])**.** *Any spacetime satisfying $T_{\mu\nu}X^\mu Y^\nu \geq 0$ for all future-directed causal vectors X, Y that contains a MOTS cannot be geodesically complete, and in particular contains a singularity in the region bounded by the MOTS.*

The following, then, is the best known result governing the possible "shapes" a black hole could be. It is a generalisation of Hawking's famous black hole uniqueness theorem; see Chapter 9 of [HE73].

Theorem 1.2.3 (Hawking)**.** *Let M^{n+1}, V^n, and Σ^{n-1} be as above, and suppose M satisfies the conditions of Theorem 1.2.2. Then Σ admits a metric of positive scalar curvature unless the following conditions hold: Σ is Ricci-flat and χ_+ and $T_{\mu\nu}u^\mu l_+^\nu$ are both identically 0.*

A proof of this theorem can be found in §7.4 of [Hor12]. We will not discuss it here and instead address its direct implication: except in some special cases, if we wish to classify the possible topologies of higher dimensional black holes, a good place to start would be to look for manifolds which admit metrics of positive scalar curvature. Hence we come to with the following question:

Question 1.2.4. *Which compact, orientable manifolds, up to diffeomorphism, admit metrics of positive scalar curvature?*

Note that in our 4-dimensional spacetime, this question is answered by the Gauss-Bonnet theorem: among surfaces, only S^2 admits a metric positive scalar curvature. In fact, Hawking used this in his proof, showing all 2-dimensional MOTS's are topologically spherical.

In other dimensions, however, we do not have the Gauss-Bonnet theorem, and so we must find other ways of relating curvature and topology. The remainder of this paper is thus devoted to investigating some of the known topological obstructions to positive scalar curvature.. There are two main approaches to doing this: the *Dirac operator approach* and the *minimal hypersurfaces approach*. In the following sections, we'll examine each of them and see some of the results that have come from each.

2 Spin Geometry and Index Theory

We begin this section by discussing the central argument that the Dirac operator approach relies on when relating curvature and topology. It is a method known as the *Bochner technique*, and it was originally developed in the context of differential forms.

Recall that a k-form ω on a Riemannian manifold M^n is *harmonic* if it is both closed and co-closed, that is, if

$$\mathrm{d}\omega = 0 \quad \text{and} \quad \mathrm{d}^*\omega := (-1)^{nk+k+1} * \mathrm{d} * \omega = 0$$

where $*$ indicates the Hodge star. We then have the Hodge Theorem, which states that there is a unique harmonic form in each de Rham cohomology group for a compact orientable manifold. We therefore expect that, by studying the harmonic forms on such a manifold, we should be able to deduce something about its topology.

What Bochner realised was that this relationship could be exploited to give geometric information by comparing two different Laplacians for 1-forms on M. One of these is the Hodge Laplacian, $\mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$. The other is obtained by viewing the Levi-Civita connection ∇ acting on 1-forms on M as a map:

$$\nabla \colon \Gamma(T^*M) \to \Gamma(T^*M) \otimes \Gamma(T^*M)$$

In this context, one can define its adjoint ∇^*; this lets us define another Laplacian called the *connection Laplacian*, $\nabla^*\nabla$. Bochner's formula then reads:

$$(\mathrm{dd}^* + \mathrm{d}^*\mathrm{d})\omega = \nabla^*\nabla\omega + \mathrm{Ric}(\omega)$$

where Ric here is the Ricci curvature operator. This formula implies that the Ricci curvature of M can be related through harmonic 1-forms to the topology of M. For more information on this argument, we recommend Chapter 7 of [Pet06].

The goal of the following section is to produce an analogous result that relates the topology of M to its *scalar* curvature. To do this, we will formally define the connection Laplacian for general vector bundles and replace each of the other components of the above equation - the Hodge Laplacian with the square of the Dirac operator and 1-forms with spinors. Finally, in lieu of the Hodge Theorem, we will have another result that relates "harmonic" spinors to the topology of the underlying manifold: the Atiyah-Singer index theorem. Now with a clear picture of our goal, let us dive in.

2.1 Bundles and Spin Groups

To begin, we need a bit of background on the theory of fibre bundles over manifolds. Our first definition is of a principal bundle.

Definition 2.1.1. A fibre bundle $F \hookrightarrow P \to M$ is a *principal G-bundle* for a Lie group G if there exists a free, transitive, continuous right action $\mu\colon G \times P \to P$ that preserves the fibres of P and such that $\mu(g, p) = p.g$ is a homeomorphism when restricted to each fibre.

One such bundle is the *oriented orthonormal frame bundle* of a Riemannian manifold (M, g). This bundle, which we denote $\mathcal{F}(M)$, is a principal $\mathrm{SO}(n)$-bundle over M whose fibre at a point p is the vector space of choices of positively oriented frames at p that are orthonormal with respect to g. $\mathrm{SO}(n)$ acts on the right on this space by change of basis transformations; for $g \in \mathrm{SO}(n)$ and f a frame:

$$f.g = f \circ g\colon\ \mathbb{R}^n \to T_pM$$

One starting point for spin geometry is the observation that $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}_2$ for $n \geq 3$. (The reader is invited to check this by performing the so-called belt trick; see [Sta10].) We deduce that there exists some other Lie group which is the universal cover of $\mathrm{SO}(n)$ with a twofold covering map. We call this group $\mathrm{Spin}(n)$. Hence we ask whether, given M, there exists a principal $\mathrm{Spin}(n)$-bundle over M with a bundle map to $\mathcal{F}(M)$ that induces the covering map $\zeta\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ on each of the fibres. The answer not being immediately obvious, we make it a definition:

Definition 2.1.2. We say that a manifold M is *spin* if there exists a principal $\mathrm{Spin}(n)$-bundle $\tilde{\mathcal{F}}(M)$ over M together with a bundle map $\xi\colon \tilde{\mathcal{F}}(M) \to \mathcal{F}(M)$ fixing the base such that ξ restricted to each fibre is the covering map $\zeta\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$. The bundle $\tilde{\mathcal{F}}(M)$ is called the *spin frame bundle* of M.

It turns out that not all manifolds are spin, but the obstruction to being spin is topological, involving characteristic classes of the tangent bundle.

Theorem 2.1.3 (Haefliger 1956)**.** *A compact orientable manifold M is spin if and only if the second Stiefel-Whitney class of its tangent bundle vanishes.*

A proof can be found in Chapter II, §1 of [LM89]. Note that the first Stiefel-Whitney class of TM vanishes if and only if M is orientable. Hence this theorem tells us that being spin is somehow analogous to being "higher-dimensional-orientable". All 3-manifolds are spin; the easiest examples of non-spin manifolds are in dimension 4, where S^4 is spin but $\mathbb{CP}^2$ is not.

Another relevant construction is called the *associated bundle construction* for a principal bundle, defined below.

Definition 2.1.4. Let $\pi\colon P \to M$ be a principal G-bundle over a manifold M. Let V be a space with $\rho\colon G \to \mathrm{Homeo}(V)$ a left action of G on V. Then G admits a left action on the space $P \times V$ by:

$$g \cdot (p, v) = (p.g^{-1}, \rho(g)(v))$$

The quotient of $P \times V$ by this action is then a fibre bundle over M with fibre V and structure group G. We call this bundle $E \to M$ the *bundle associated to P by ρ* and denote it as $E = P \times_\rho V$.

Heuristically, the associated bundle construction preserves the global structure of the original principal bundle but "glues in" copies of the new space V as the fibres. In particular, the new fibres retain an action of the structure group.

In our case, V will itself be a vector space, and the action ρ will be a *representation* of G on V; that is, a homomorphism $G \to \mathrm{GL}(V)$. For example:

Example 2.1.5. Let $\rho\colon \mathrm{SO}(n) \to \mathrm{GL}(n, \mathbb{R}) \subset \mathrm{Homeo}(\mathbb{R}^n)$ be the inclusion map, so that $\mathrm{SO}(n)$ acts on $\mathbb{R}^n$ by linear transformations. Then:

$$\mathcal{F}(M) \times_\rho \mathbb{R}^n \cong TM$$

We can modify this to get other familiar bundles. Let ρ^* be the dual representation; that is, $\rho^*(g) = \rho(g^{-1})^T$, and let λ be the induced exterior power representation. Then:

$$\mathcal{F}(M) \times_{\rho^*} \mathbb{R}^n \cong T^*M$$
$$\mathcal{F}(M) \times_\lambda \Lambda^* \mathbb{R}^n \cong \Lambda^*(T^*M)$$

where $\Lambda^*(T^*M)$ is the bundle of differential forms on M.

We want to use this intuition to construct a new bundle on M with the structure of the spin frame bundle on M, but one which has a different fibre analogous to the tangent space. To do this, we need to construct suitable representations of $\mathrm{Spin}(n)$ on an appropriate vector space. In the following section, we explain these representations and use them to construct spinor bundles following the associated bundle construction.

2.2 Constructions of Spinors

We begin with the definition of a Clifford algebra, a necessary structure for defining spinors.

Definition 2.2.1. Let V be a vector space over a field k (of characteristic not 2) and let q be a quadratic form on V. Then the Clifford algebra $\mathrm{Cliff}(V, q)$ is the quotient of the tensor algebra $\bigoplus_{j=0}^{\infty}(\bigotimes_{i=0}^{j} V)$ by the two-sided ideal generated by elements of the form $v \otimes v - q(v)1$, where 1 is the multiplicative identity of the tensor algebra.

For intuition, note that $\mathrm{Cliff}(V, q)$ can be considered to be a generalisation of the exterior algebra. As vector spaces, in fact, $\mathrm{Cliff}(V, q)$ is isomorphic to $\Lambda^*(V) = \bigoplus_{k=0}^{\infty} \Lambda^k(V)$. This isomorphism extends to an algebra isomorphism if and only if q is the 0 form.

Clifford algebras are naturally $\mathbb{Z}_2$-graded; they decompose as:

$$\mathrm{Cliff}(V, q) = \mathrm{Cliff}^0(V, q) \oplus \mathrm{Cliff}^1(V, q)$$

with $\mathrm{Cliff}^0(V, q)$ a subalgebra. The grading is obtained as follows: consider the map $v \mapsto -v$ on V. As this map preserves the form $q(v)$, it extends to an algebra automorphism $\alpha\colon \mathrm{Cliff}(V, q) \to \mathrm{Cliff}(V, q)$. One can check that $\alpha^2 = \mathrm{Id}$, letting us define:

$$\mathrm{Cliff}^i(V, q) := \{c \in \mathrm{Cliff}(V, q) \mid \alpha(c) = (-1)^i c\}$$

Again for intuition, this grading splits $\mathrm{Cliff}(V, 0) \cong \Lambda^*(V)$ as :

$$\Lambda^*(V) = \Lambda^{\mathrm{even}}(V) \oplus \Lambda^{\mathrm{odd}}(V)$$

For the uses of Clifford algebras $\mathrm{Cliff}(V, q)$ in this paper, we will in fact only consider the cases where $V \cong \mathbb{R}^n$ or $\mathbb{C}^n$, and where q is given by:

$$q(x) = x_1^2 + \cdots + x_n^2$$

To simplify notation, we will from here denote the real (complex) Clifford algebra with positive definite quadratic form in dimension n as $\mathrm{Cliff}(n)$ ($\mathrm{Cliff}_{\mathbb{C}}(n)$). For more information on general Clifford algebras, see Chapter I of [LM89].

For intuition, we provide a classification of $\mathrm{Cliff}(n)$ and $\mathrm{Cliff}_{\mathbb{C}}(n)$ for n up to 8 in Table 1. The notation $M_m(k)$ indicates the algebra of $m \times m$ matrices with entries in the division algebra k. From this table we can actually classify $\mathrm{Cliff}(n)$ for all n, as it is a consequence of Bott periodicity that $\mathrm{Cliff}(8k+r) \cong \mathrm{Cliff}(8) \otimes \mathrm{Cliff}(r)$, and the same for $\mathrm{Cliff}_{\mathbb{C}}(n)$; again see [LM89] for a proof.

n	$\mathrm{Cliff}(n)$	$\mathrm{Cliff}_{\mathbb{C}}(n)$
1	$\mathbb{C}$	$\mathbb{C} \oplus \mathbb{C}$
2	$\mathbb{H}$	$M_2(\mathbb{C})$
3	$\mathbb{H} \oplus \mathbb{H}$	$M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$
4	$M_2(\mathbb{H})$	$M_4(\mathbb{C})$
5	$M_4(\mathbb{C})$	$M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$
6	$M_8(\mathbb{R})$	$M_8(\mathbb{C})$
7	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$	$M_8(\mathbb{C}) \oplus M_8(\mathbb{C})$
8	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{C})$

Table 1: Low-dimensional real and complex Clifford algebras with positive definite quadratic form, adapted from [LM89].

There are a lot of symmetries hidden in this classification. For instance, it turns out that $\mathrm{Cliff}_{\mathbb{C}}(n) \cong \mathrm{Cliff}(n) \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathrm{Cliff}^0(n) \cong \mathrm{Cliff}(n-1)$. These two facts allow for the computation of almost this entire table; for proofs, see Chapter I of [LM89].

We have bothered with all of this because:

Proposition 2.2.2. *Spin(n) embeds as a subgroup of the group of units Cliff$^\times(n)$ of Cliff(n).*

See Chapter I, §2 of [LM89] for a full proof; the idea is to use the adjoint representation of $\mathrm{Cliff}^\times(n)$ on $\mathrm{Aut}(\mathrm{Cliff}(n))$ to construct a representation of the subgroup $G < \mathrm{Cliff}^\times(n)$ given by:

$$G = \{v_1 v_2 \dots v_k \mid v_i \in V, q(v_i) = 1\} \cap \mathrm{Cliff}^0(n)$$

on $\mathbb{R}^n$ so that the corresponding map to $\mathrm{SO}(n)$ is a degree 2 covering map.

Recall that to construct a bundle associated to a principal $\mathrm{Spin}(n)$-bundle, we need a representation of $\mathrm{Spin}(n)$. However, some representations of $\mathrm{Spin}(n)$ are simply lifts of representations of $\mathrm{SO}(n)$, which produce $\mathcal{F}(M)$-associated bundles. We have introduced Clifford algebras because the *other* representations, the so-called the *spin representations*, are those that come from restricting certain representations of $\text{Cliff}(n)$ (or $\text{Cliff}_{\mathbb{C}}(n)$) to $\text{Spin}(n)$. Thus Clifford algebras are themselves essential to the study of bundles associated to $\tilde{\mathcal{F}}(M)$, and they provide structure for understanding the action of spin representations. In fact, we can now give a definition for a spinor bundle over M.

Definition 2.2.3. A *spinor bundle* over M is a bundle $\mathcal{S}(M)$ obtained by:

$$\mathcal{S}(M) = \tilde{\mathcal{F}}(M) \times_\rho V$$

where V is a vector space and ρ is a spin representation. A section of $\mathcal{S}(M)$ is called a spinor, and $\mathcal{S}(M)$ is said to be a real (complex) spinor bundle if V is real (complex) as a vector space, and *irreducible* if ρ is an irreducible representation.

In brief, the particular representations of $\text{Cliff}(n)$ that we restrict to form spin representations come from the matrix algebra structures of $\text{Cliff}_{\mathbb{C}}(n)$. The algebras $M_m(k)$ have natural faithful representations on k^{2^m}, and so we can use these to obtain representations of $\text{Cliff}_{\mathbb{C}}(n)$ on $\mathbb{C}^{\lceil n/2 \rceil}$. To avoid getting bogged down in a lot of algebra, we refer the reader to [LM89] for the precise construction and restriction to $\text{Cliff}(n)$; we will only need the information in Table 2, adapted from the same text. Here, v_n is the number of equivalence classes of irreducible representations, and we have also listed the dimensions of these representations.

n	$\text{Cliff}(n)$	$v_n^{\mathbb{R}}$	$\dim_n^{\mathbb{R}}$	$\text{Cliff}_{\mathbb{C}}(n)$	$v_n^{\mathbb{C}}$	$\dim_n^{\mathbb{C}}$
1	$\mathbb{C}$	1	2	$\mathbb{C} \oplus \mathbb{C}$	2	1
2	$\mathbb{H}$	1	4	$M_2(\mathbb{C})$	1	2
3	$\mathbb{H} \oplus \mathbb{H}$	2	4	$M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$	2	2
4	$M_2(\mathbb{H})$	1	8	$M_4(\mathbb{C})$	1	4
5	$M_4(\mathbb{C})$	1	8	$M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$	2	4
6	$M_8(\mathbb{R})$	1	8	$M_8(\mathbb{C})$	1	8
7	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$	2	8	$M_8(\mathbb{C}) \oplus M_8(\mathbb{C})$	2	8
8	$M_{16}(\mathbb{R})$	1	16	$M_{16}(\mathbb{C})$	1	16

Table 2: Number and dimension of irreducible, faithful representations of Clifford algebras, adapted from [LM89].

It will shortly be useful to have slightly different definition for spinors, which we now describe. First note that a representation ρ of a Clifford algebra $\text{Cliff}(n)$ on a vector space V defines a module over the ring structure of $\text{Cliff}(n)$ with underlying group V. In fact, the representation ρ gives the multiplication map $\text{Cliff}(n) \times V \to V$ by:

$$c \cdot v = \rho(c)(v)$$

We call this multiplication the *Clifford multiplication*. Note that the conditions that the Clifford multiplication must satisfy to be a module multiplication are directly implied by the requirement that ρ is $\mathbb{R}$-linear as a representation. That said, we can define a *Clifford module* to be a module over the ring structure of a Clifford algebra obtained from a representation in this manner. We say that a Clifford module is *irreducible* if it is obtained from an irreducible representation of $\text{Cliff}(n)$.

This observation allows us to more directly use the representations of Clifford algebras to understand representations of spin groups. In fact, if we have a representation of $\text{Cliff}(n)$ on V, then the inclusion $\text{Spin}(n) \hookrightarrow \text{Cliff}(n)$ induces a representation of $\text{Spin}(n)$ on V. Let $\mu\colon \text{Spin}(n) \to \text{Aut}(\text{Cliff}(n))$ denote the action of $\text{Spin}(n)$ on $\text{Cliff}(n)$ by left multiplication: that is, $\mu(g)(c) = g \cdot c$. This lets us give an alternate definition of a spinor bundle:

Definition 2.2.4. Let N be a left module over $\text{Cliff}(n)$ obtained from a representation $\rho\colon \text{Cliff}(n) \to \text{GL}(V)$ for V a real (complex) vector space. Then the associated real (complex) *spinor bundle* $\mathcal{S}(M)$ over M is the bundle:

$$\mathcal{S}(M) = \tilde{\mathcal{F}}(M) \times_\mu V$$

All of the above holds for $\text{Cliff}_{\mathbb{C}}(n)$ as well, as $\text{Spin}(n) \subset \text{Cliff}(n) \subset \text{Cliff}_{\mathbb{C}}(n)$.

Given this definition, it would seem that we could view spinor bundles as bundles of modules in some sense. To formalise this, we make the following definition.

Definition 2.2.5. Let (M, g) be a Riemannian manifold. Then the *Clifford bundle* $\text{Cliff}(M)$ over M is the vector bundle of real Clifford algebras such that at each point $p \in M$, $\text{Cliff}(M)_p = \text{Cliff}(T_pM, g_p)$, the Clifford algebra on T_pM with the quadratic form given by the metric.

Note because g_p is positive definite at each p in M, each fibre $\text{Cliff}(M)_p$ is isomorphic as an $\mathbb{R}$-algebra to $\text{Cliff}(n)$ (just pick an orthonormal basis for T_pM). This gives us spinors as bundles of modules.

Proposition 2.2.6. *Let $\mathcal{S}(M)$ be a real spinor bundle over M. Then the sections of $\mathcal{S}(M)$ form a module over the sections of Cliff(M).*

A proof can be found in Chapter II, §3 of [LM89]; it is a diagram chase.

We say that two spinor bundles are equivalent if they are isomorphic as modules over Cliff(M). In a moment, we will give data on the number of equivalence classes of spinor bundles by dimension. But before we do, we note that one important feature exhibited by some Clifford modules is a $\mathbb{Z}_2$-*grading*, that is, a decomposition $V \cong V^0 \oplus V^1$ such that:

$$\mathrm{Cliff}^i(n) \cdot V^i \subseteq V^{i+1}$$

with indices taken mod 2. When a grading can be defined globally on a bundle of modules over Cliff(M), by Proposition 2.2.6, we can say that spinor bundles themselves are graded or ungraded. Graded spinor bundles are written:

$$\mathcal{S}(M) \cong \mathcal{S}^+(M) \oplus \mathcal{S}^-(M)$$

In this case, Clifford multiplication interchanges these two subbundles, which we call *chiral spinor bundles*. We now count the equivalence classes of spinor bundles in Table 3 below, as it can be deduced from examining Table 2.

	Real		Complex	
n mod 8	Ungraded	Graded	Ungraded	Graded
1	1	1	2	1
2	1	1	1	2
3	2	1	2	1
4	1	2	1	2
5	1	1	2	1
6	1	1	1	2
7	2	1	2	1
8	1	2	1	2

Table 3: Number of equivalence classes of real, complex, graded, and ungraded spinor bundles over a connected spin manifold of dimension n, adapted from [LM89].

Finally, we put an inner product structure on spinors that respects the Clifford multiplication.

Proposition 2.2.7. *Let $\mathcal{S}(M)$ be a spinor bundle with typical fibre V. Then there exists an k-linear inner product $\langle\ ,\ \rangle$ on V such that for all $v, w \in V$ and unit vectors $e_i \in T_pM$,*

$$\langle e_i \cdot v, e_i \cdot w \rangle = \langle v, w \rangle$$

and for all $c \in$ Cliff(n),

$$\langle c \cdot v, w \rangle = -\langle v, c \cdot w \rangle$$

for $k = \mathbb{R}$ or $\mathbb{C}$ (if $\mathcal{S}(M)$ is real or complex).

A proof can be found in Chapter I, §5 of [LM89]; one picks a k-linear inner product on V and then averages it over an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM. A smooth global choice of such an inner product is equivalent to choosing a bundle metric on $\mathcal{S}(M)$ that is compatible with the metric g on M. This gives rise to a global inner product $(\ ,\ )$ for compactly supported sections σ, τ of $\mathcal{S}(M)$ defined by:

$$(\sigma, \tau) = \int_M \langle \sigma, \tau \rangle_p$$

This inner product structure is essential for later applications, notably the Dirac operator.

2.3 Connections on Spinor Bundles

Having now defined spinors, we want to understand their geometry and curvature. The first step to achieving this is to define a connection on spinor bundles, so that we may take covariant derivatives of spinors. Given a covariant derivative rule, we can then define curvature and other related operators.

The full construction of the spin connection can be found in [LM89], but we omit it here for space and clarity. Roughly, it follows three steps. First, we use the Levi-Civita connection on the tangent bundle TM to define a particular $\mathfrak{so}(n)$-valued 1-form ω called the *connection 1-form* on the oriented orthonormal frame bundle $\mathcal{F}(M)$ viewed as a manifold itself. The kernel of ω is a distribution of "horizontal" subspaces on $T\mathcal{F}(M)$ induced by the Levi-Civita connection on TM. Since $\mathfrak{so}(n)$ is the space of real, skew-symmetric $n \times n$ matrices, we can write ω locally as:

$$\omega = \sum_{i<j} \omega_{ij}\ e_i \wedge e_j$$

where $e_i \wedge e_j$ is the elementary skew-symmetric (i,j)-matrix and ω_{ij} are $\mathbb{R}$-valued 1-forms. (We use e_i for these because the matrix $e_i \wedge e_j \in \mathfrak{so}(n)$ corresponds to the element $\frac{1}{2} e_i \cdot e_j \in \mathfrak{spin}(n)$, which is isomorphic to $\mathfrak{so}(n)$.) We can then lift ω to another connection 1-form $\tilde{\omega}$ on $\tilde{\mathcal{F}}(M)$ via the covering map with the same decomposition:

$$\tilde{\omega} = \sum_{i<j} \tilde{\omega}_{ij} \; e_i \wedge e_j$$

Finally, we use $\tilde{\omega}$ to define a metric covariant derivative rule on an associated spinor bundle $\mathcal{S}(M)$ using the associated bundle construction. The end result is the following:

Theorem 2.3.1. *Let ω be the connection 1-form on $\mathcal{F}(M)$ induced by the Levi-Civita connection, $\tilde{\omega}$ be its lift to $\tilde{\mathcal{F}}(M)$, and $\mathcal{S}(M)$ be any spinor bundle associated to $\tilde{\mathcal{F}}(M)$. Then the induced covariant derivative $\nabla \colon \Gamma(\mathcal{S}(M)) \to \Gamma(T^*M) \otimes \Gamma(\mathcal{S}(M))$ is given for a local orthonormal frame $\{e_1, \ldots, e_n\}$ by:*

$$\nabla_{e_i} = \partial_{e_i} + \frac{1}{2} \sum_{i<j} \tilde{\omega}_{ij} \otimes (e_i \cdot e_j \cdot)$$

where ∂_{e_i} acts on the components of a spinor with respect to an orthonormal frame for $\mathcal{S}(M)$ with respect to the bundle metric $\langle \, , \, \rangle$.

Like all metric covariant derivatives, ∇ respects a Leibniz rule for $f \in C^\infty(M)$ and $\sigma \in \mathcal{S}(M)$:

$$\nabla(f\sigma) = df \otimes \sigma + f \nabla \sigma$$

and the fibre metric:

$$X\langle \sigma, \tau \rangle = \langle \nabla_X \sigma, \tau \rangle + \langle \sigma, \nabla_X \tau \rangle$$

Thus the covariant derivative for spinors consists of nothing more than data from the Levi-Civita connection and Clifford multiplication.

As for curvature, for $X, Y \in \Gamma(TM)$ the *curvature operator* $R^E_{X,Y} \colon \Gamma(E) \to \Gamma(E)$ for any vector bundle E equipped with a metric connection is:

$$R^E_{X,Y} = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

Thus using Theorem 2.3.1, we can compute:

Theorem 2.3.2. *Let $X, Y \in \Gamma(TM)$ and $\{e_1, \cdots, e_n\}$ be a local orthonormal frame. Then the curvature transformation $R^{\mathcal{S}}_{X,Y} \colon \Gamma(\mathcal{S}(M)) \to \Gamma(\mathcal{S}(M))$ is given by:*

$$R^{\mathcal{S}}_{X,Y}(\sigma) = \frac{1}{2} \sum_{i<j} g(R(X,Y)e_i, e_j) \; e_i \cdot e_j \cdot \sigma$$

where $R(X,Y)$ is the Riemann curvature operator for (M,g).

Now as with the Bochner formula, we wish to compare two different generalisations of the Laplacian. The first generalisation is the *connection Laplacian*, defined as follows.

Definition 2.3.3. The *connection Laplacian* $\nabla^*\nabla\colon \Gamma(\mathcal{S}(M)) \to \Gamma(\mathcal{S}(M))$ is defined such that:

$$(\nabla\sigma, \nabla\sigma) = (\nabla^*\nabla\sigma, \sigma)$$

Equivalently,

$$\nabla^*\nabla\sigma := -\mathrm{tr}((X, Y) \mapsto \nabla_X\nabla_Y\sigma - \nabla_{\nabla_X Y}\sigma)$$

The other generalisation we make uses the Dirac operator, which we devote the next section to.

2.4 Dirac Operators and the Lichnerowicz Formula

In our discussion of Dirac operators, we follow Chapter II, §5 of [LM89].

Definition 2.4.1. Let $\mathcal{S}(M)$ be a spinor bundle on M. The *Dirac operator* associated to $\mathcal{S}(M)$ is the first order differential operator whose action on a spinor $\sigma \in \Gamma(\mathcal{S}(M))$ is given in local coordinates by:

$$\not{D}\sigma = \sum_{j=1}^{n} e_j \cdot \nabla_{e_j}\sigma$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame for TM.

An important case is when $\mathcal{S}(M)$ is $\mathbb{Z}_2$-graded. In this case, we can restrict $\not{D}$ to each subbundle $\mathcal{S}^+(M)$ and $\mathcal{S}^-(M)$ to obtain *chiral* Dirac operators $\not{D}^+$ and $\not{D}^-$. Because Clifford multiplication by e_i interchanges $\mathcal{S}^+(M)$ and $\mathcal{S}^-(M)$, we have:

$$\not{D}^+\mathcal{S}^+(M) \subseteq \mathcal{S}^-(M) \qquad \text{and} \qquad \not{D}^-\mathcal{S}^-(M) \subseteq \mathcal{S}^+(M)$$

There are a few comments to make on Dirac operators in the context of the theory of differential operators.

Definition 2.4.2. Let E and F be k-vector bundles on a manifold M and let $D\colon \Gamma(E) \to \Gamma(F)$ be an order m differential operator. The *principal symbol* of D is a map $\sigma\colon T^*M \otimes E \to F$ such that for all $\xi \in T_p^*M$, $\sigma_\xi(D) \in \mathrm{Hom}_k(E_p, F_p)$ is a linear map defined in local coordinates as follows. Suppose we are given expressions in local coordinates for D and ξ below:

$$D = \sum_{|\alpha|\leq m} A_\alpha(x)\frac{\partial^{|\alpha|}}{\partial x^\alpha} \qquad \text{and} \qquad \xi = \sum_k \xi_k dx^k$$

where we have used multi-index notation so that $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Then $\sigma_\xi(D)$ is given by:

$$\sigma_\xi(D) = i^m \sum_{|\alpha|=m} A_\alpha(x)\xi^\alpha$$

where ξ^α is $\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$.

A differential operator is said to be *elliptic* if its principal symbol $\sigma_\xi(D)$ is a k-linear isomorphism for all ξ.

Proposition 2.4.3. *$\not{D}$ and $\not{D}^2$ are both elliptic.*

Proof. We compute the principal symbol in local coordinates at p, using the Riemannian metric to identify T_p^*M with T_pM (hence viewing ξ as $\xi^\sharp \in T_pM$). For $\not{D}$, note that if we choose geodesic normal coordinates $\{e_1, \ldots, e_n\}$ with the origin at p, then $\nabla_{e_j}|_0 = \frac{\partial}{\partial x_j}|_0 +$ zero order terms. Hence $\not{D}$ can be written at p as:

$$\begin{aligned}\not{D} &= \sum_{|\alpha|\leq 1} A_\alpha(x)\frac{\partial}{\partial x_j} \\ &= \sum_{j=1}^n e_j \cdot \frac{\partial}{\partial x_j} + 0 \text{ order terms}\end{aligned}$$

Therefore the symbol for $\xi \in T_p^*M$ is:

$$\begin{aligned}\sigma_\xi(\not{D}) &= i\sum_{j=1}^n e_j\xi_j\cdot \\ &= i\xi\cdot\end{aligned}$$

where this notation indicates Clifford multiplication by $i\xi^\sharp$, which is an isomorphism for $\xi \neq 0$ since Clifford multiplication is really an action on the space of spinors.

For $\not{D}^2$, we note that $\sigma_\xi(\not{D}^2) = \sigma_\xi(\not{D}) \circ \sigma_\xi(\not{D})$. Hence:

$$\sigma_\xi(\not{D}^2) = i\xi \cdot (i\xi\cdot) = -g_p(\xi, \xi)$$

where now the multiplication is just normal scalar multiplication, which is invertible for $\xi \neq 0$ since g is positive definite. $\square$

Note that this implies that the restrictions $\not{D}^+$ and $\not{D}^-$ are also elliptic.

There is a rich theory for elliptic operators which we can now draw upon. For instance, elliptic operators always have finite dimensional kernel and cokernel. This lets us define the analytic index:

Definition 2.4.4. The *analytic index* of an elliptic operator D is

$$\text{ind}_a = \dim \ker D - \dim \text{coker}\, D$$

As suggested by the name "index theorem", we will be interested in the indices of Dirac operators. However, note the following:

Proposition 2.4.5. *The Dirac operator associated to a spinor bundle $\mathcal{S}(M)$ is formally self-adjoint with respect to $(\ ,\ )$.*

Proof. Fix $p \in M$ and let $\{e_1, \dots, e_n\}$ be a frame obtained from geodesic normal coordinates near p. Then:

$$\begin{aligned}
\langle \not{D}\sigma, \tau\rangle_p &= \sum_i \langle e_i \cdot \nabla_{e_i}\sigma, \tau\rangle_p \\
&= -\sum_i \langle \nabla_{e_i}\sigma, e_i \cdot \tau\rangle_p \\
&= -\sum_i \big(e_i\langle \sigma, e_i \cdot \tau\rangle_p - \langle \sigma, \nabla_{e_i}(e_i \cdot \tau)\rangle_p\big) \\
&= -\sum_i \big(e_i\langle \sigma, e_i \cdot \tau\rangle_p - \langle \sigma, (\nabla_{e_i}e_i) \cdot \tau\rangle_p - \langle \sigma, e_i \cdot \nabla_{e_i}\tau\rangle_p\big) \\
&= \langle \sigma, \not{D}\tau\rangle_p - \sum_i e_i\langle \sigma, e_i \cdot \tau\rangle_p
\end{aligned}$$

Let V be the vector field such that for all $W \in T_pM$,

$$\langle V, W\rangle_p = -\langle \sigma, W \cdot \tau\rangle_p$$

Then:

$$-\sum_i e_i\langle \sigma, e_i \cdot \sigma\rangle_p = \sum_i e_i\langle V, e_i\rangle_p = \text{div}(V)_p$$

Hence,

$$\langle \not{D}\sigma, \tau\rangle_p = \langle \sigma, \not{D}\tau\rangle_p + \text{div}(V)_p$$

So by the divergence theorem, integrating over a compact M gives us:

$$(\not{D}\sigma, \tau) = (\sigma, \not{D}\tau)$$

□

Thus since $\dim \operatorname{coker} \not{D} = \dim \ker \not{D}^* = \dim \ker \not{D}$, $\operatorname{ind}_a(\not{D}) = 0$ for any Dirac operator. We say a spinor σ is *harmonic* if $\sigma \in \ker \not{D}$.

We can now prove our Bochner-style formula, following [LM89]:

Theorem 2.4.6 (Lichnerowicz)**.** *Let $\mathcal{S}(M)$ be any spinor bundle on a compact, orientable Riemannian M, σ be a section of $\mathcal{S}(M)$, and $\not{D}$ be the associated Dirac operator. Then:*

$$\not{D}^2\sigma = \nabla^*\nabla\sigma + \frac{S}{4}\sigma$$

where S is the scalar curvature of M.

Proof. Let $p \in M$ and let $\{e_1, \ldots, e_n\}$ be an orthonormal frame obtained from geodesic normal coordinates centred at p, so that $\nabla_{e_i} e_j = 0$. Then we have:

$$\begin{aligned}
\not{D}^2\sigma &= \sum_i e_i \cdot \nabla_{e_i} \left(\sum_j e_j \cdot \nabla_{e_j} \sigma \right) \\
&= \sum_{i,j} e_i \cdot e_j \cdot \nabla_{e_i} \nabla_{e_j} \sigma \\
&= \sum_i e_i \cdot e_i \cdot \nabla_{e_i} \nabla_{e_i} \sigma + \sum_{i \neq j} e_i \cdot e_j \cdot \nabla_{e_i} \nabla_{e_j} \sigma \\
&= -\sum_i \nabla_{e_i} \nabla_{e_i} \sigma + \sum_{i \neq j} e_i \cdot e_j \cdot \nabla_{e_i} \nabla_{e_j} \sigma
\end{aligned}$$

where we have used the Clifford multiplication rule for normalised vectors $e_i \cdot e_i = -1$. So by the definition of the connection Laplacian, this gives us:

$$\not{D}^2\sigma = \nabla^*\nabla\sigma + \sum_{i \neq j} e_i \cdot e_j \cdot \nabla_{e_i} \nabla_{e_j} \sigma$$

We compute the last term separately:

$$\begin{aligned}
\sum_{i\neq j} e_i\cdot e_j\cdot \nabla_{e_i}\nabla_{e_j}\sigma &= \frac{1}{2}\sum_{i\neq j} e_i\cdot e_j\cdot \nabla_{e_i}\nabla_{e_j}\sigma + \frac{1}{2}\sum_{i\neq j} e_i\cdot e_j\cdot \nabla_{e_i}\nabla_{e_j}\sigma \\
&= \frac{1}{2}\sum_{i\neq j} e_i\cdot e_j\cdot \nabla_{e_i}\nabla_{e_j}\sigma - \frac{1}{2}\sum_{i\neq j} e_j\cdot e_i\cdot \nabla_{e_i}\nabla_{e_j}\sigma \\
&= \frac{1}{2}\sum_{i\neq j} e_i\cdot e_j\cdot \nabla_{e_i}\nabla_{e_j}\sigma - \frac{1}{2}\sum_{i\neq j} e_i\cdot e_j\cdot \nabla_{e_j}\nabla_{e_i}\sigma \\
&= \frac{1}{2}\sum_{i\neq j} e_i\cdot e_j\cdot (\nabla_{e_i}\nabla_{e_j}\sigma - \nabla_{e_j}\nabla_{e_i}\sigma) \\
&= \frac{1}{2}\sum_{i,j} e_i\cdot e_j\cdot R^{\mathbb{S}}_{e_i,e_j}(\sigma)
\end{aligned}$$

Note that the last equivalence holds because $[e_i, e_j] = 0$ for coordinate vector fields and we are able replace $i \neq j$ in the sum with i, j because the terms with $i = j$ reduce to 0. But we know from Theorem 2.3.2 what $R^{\mathbb{S}}_{e_i,e_j}(\sigma)$ is:

$$\begin{aligned}
\frac{1}{2}\sum_{i,j} e_i\cdot e_j\cdot R^{\mathbb{S}}_{e_i,e_j}(\sigma) &= \frac{1}{2}\sum_{i,j} e_i\cdot e_j\cdot \left(\frac{1}{4}\sum_{k,l} e_k\cdot e_l\cdot g(R(e_i,e_j)e_k,e_l)\sigma\right) \\
&= \frac{1}{8}\sum_{i,j,k,l} R_{ijkl}\ e_i\cdot e_j\cdot e_k\cdot e_l\cdot\sigma
\end{aligned}$$

We then reduce using the symmetries of the Riemann curvature tensor. Note that if i, j, k are all distinct, then:

$$e_i\cdot e_j\cdot e_k = e_j\cdot e_k\cdot e_i = e_k\cdot e_i\cdot e_j$$

and so we we can use the first Bianchi identity to get:

$$\sum_{l}\sum_{\substack{i,j,k\\ \text{distinct}}} (R_{ijkl} + R_{jkil} + R_{kijl}) e_i\cdot e_j\cdot e_k\cdot\sigma = 0$$

Thus we are left with the remaining terms:

$$\begin{aligned}
\frac{1}{8}\sum_{i,j,k,l} R_{ijkl}\ e_i\cdot e_j\cdot e_k\cdot e_l\cdot\sigma &= \frac{1}{8}\sum_{i,k,l}(R_{ikkl}\ e_i\cdot e_k\cdot e_k\cdot e_l + R_{kikl}\ e_k\cdot e_i\cdot e_k\cdot e_l)\cdot\sigma \\
&= -\frac{1}{4}\sum_{i,k,l} R_{ikkl}\ e_i\cdot e_k\cdot e_k\cdot e_l\cdot\sigma
\end{aligned}$$

where we have used $R_{kikl} = -R_{ikkl}$ and $e_k \cdot e_i = -e_i \cdot e_k$. Then since $e_k \cdot e_k = -1$, we have:

$$\begin{aligned} -\frac{1}{4}\sum_{i,k,l} R_{ikkl}\ e_i \cdot e_k \cdot e_k \cdot e_l \cdot \sigma &= -\frac{1}{4}\sum_{i,l} \mathrm{Ric}_{il}\ e_i \cdot e_l \cdot \sigma \\ &= \frac{1}{4} S\sigma \end{aligned}$$

Substituting this back into our form for $\not{D}^2\sigma$ gives us the desired result. □

We thus obtain our first hint at an obstruction to positive scalar curvature.

Corollary 2.4.7. *Any compact spin manifold M of positive scalar curvature admits no harmonic spinors for any spinor bundle over M.*

Proof. Let M have positive scalar curvature and suppose $\sigma \in \ker \not{D}$. Integrating the Lichnerowicz formula against σ we obtain:

$$\begin{aligned} (\not{D}^2\sigma, \sigma) &= (\nabla^*\nabla\sigma, \sigma) + (\tfrac{S}{4}\sigma, \sigma) \\ 0 = (\not{D}\sigma, \not{D}\sigma) &= (\nabla\sigma, \nabla\sigma) + \frac{1}{4}(\sigma, \sigma)\int_M S \end{aligned}$$

Since $\int_M S > 0$ and $(\ ,\ )$ is positive definite, we have a contradiction. □

The reader may recall that they were promised a connection between the topology of a compact Riemannian manifold and its scalar curvature. Thus far, after assuming that our manifold M is spin, we have been able to relate its scalar curvature to its space of harmonic spinors. It remains to show, then, that there is some relationship between that space and the topology of M - this is where we use the Atiyah-Singer index theorem.

2.5 The Index Theorem and Topological Obstructions

In preparation for the index theorem, we recall a particularly useful construction of the K-theory of a manifold in terms of complexes of k-vector bundles over it. A more detailed explanation of this setting can be found in [Lan05], from which this adapted. A proof that this formulation is equivalent to the standard one in terms of the Grothendieck group of k-vector bundles under Whitney sum is in §2.6 of [Ati89].

For simplicity of notation we consider the case $k = \mathbb{C}$; however, the following holds for the $k = \mathbb{R}$ case as well.

Definition 2.5.1. A *chain complex of vector bundles* E_* is a complex

$$0 \longrightarrow E_0 \xrightarrow{\alpha_0} E_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{n-1}} E_n \longrightarrow 0$$

where each E_k is a vector bundle over M and each α_k is a vector bundle morphism with $\alpha_k \circ \alpha_{k-1} = 0$.

The *support* of a complex is the set of $p \in M$ where the chain complex above restricted to fibres at p fails to be exact. We are interested in complexes with compact support.

Finally, we say that two complexes E_* and F_* are *homotopic* if there exists a complex G_* on $M \times [0,1]$ such that $E_* \cong G_*|_{M\times\{0\}}$ and $F_* \cong G_*|_{M\times\{1\}}$.

That said, we can define $C(M)$ to be the set of equivalence classes of such complexes (equivalence taken up to compactly supported homotopies), and $C_\varnothing(M)$ to be the set of such classes with empty support. The relationship to $K(M)$ is then:

$$K(M) \cong C(M)/C_\varnothing(M)$$

For M compact, we can obtain the element in $K(M)$ corresponding to a complex via the *Euler characteristic* map:

$$\chi(E_*) = \sum_k (-1)^k [E_k]$$

where $[E_k]$ is the class of E_k in $K(M)$ and the sum is the Whitney sum.

K-theory can be used to define a map $\mathrm{ind}_t \colon K(TM) \to \mathbb{Z}$ called the *topological index* of M. (We will use this index to define the topological index of elliptic operators later on.) This map is the composition of two maps $i_!$ and $j_!^{-1}$ as shown:

$$\mathrm{ind}_t \colon K(T^*M) \xrightarrow{i_!} K(T^*\mathbb{R}^m) \xrightarrow{j_!^{-1}} K(\mathrm{pt}) \cong \mathbb{Z}$$

Here, $i_!$ is induced by the inclusion $M \hookrightarrow \mathbb{R}^m$ for some m using the Whitney embedding theorem; it is called the *Pontryagin-Thom collapse map*. The map $j_!^{-1}$ is the induced map on K-theory from the inclusion $\{\mathrm{pt}\} \hookrightarrow \mathbb{R}^m$; it is an isomorphism by the *Thom isomorphism theorem*. The details of this map can be found in §2.7 of [Ati89].

We now see how to relate topology to Dirac operators. Recall that the principal symbol σ of an operator $D \colon \Gamma(E) \to \Gamma(F)$ is a map $T^*M \otimes \mathcal{S}(M) \to \mathcal{S}(M)$. Thus it defines a complex of vector bundles over T^*M:

$$0 \longrightarrow \pi^*E \xrightarrow{\sigma(D)} \pi^*F \longrightarrow 0$$

where π is the map $T^*M \to M$. If D is elliptic, then $\sigma_\xi(D)$ is invertible for all $\xi \neq 0$, and so this complex is exact outside the zero section. Thus $\sigma(D)$ defines an element in $K(T^*M)$, which we can then apply ind_t to. This gives an integer $\mathrm{ind}_t(D)$ associated to every elliptic operator D on M, which we also call the *topological index* of D. We then observe the following celebrated theorem.

Theorem 2.5.2 (Atiyah-Singer[1]). *Let D be an elliptic differential operator acting on smooth vector bundles over a compact manifold. Then $ind_t(D) = ind_a(D)$.*

Thus, the analytic index of an elliptic operator, which is in general very difficult to compute, can be found by computing its topological index, which is usually easier to compute and naturally relates to other topological properties. (There is another way to define ind_t using characteristic classes, which is often even easier to compute than the definition we gave!) Thus, since the Dirac operator is elliptic, together with Theorem 2.4.6, we obtain a family of results on positive scalar curvature from the index theorem. The most immediate application is the following.

Corollary 2.5.3. *Let M be a compact, spin Riemannian manifold of dimension $n = 4k$. If M admits a metric of positive scalar curvature, then $\hat{A}(M) = 0$.*

Here $\hat{A}(M)$ is a topological invariant of M that can be defined for manifolds of dimension $n = 4k$ called the $\hat{A}$-genus. It is the genus of a multiplicative sequence in the Pontryagin classes of M, making it fairly easy to compute, especially given the properties:

i. $\hat{A}(M \times N) = \hat{A}(M)\hat{A}(N)$,

ii. $\hat{A}(M) = 0$ if there exists W such that $M = \partial W$, and

iii. For compact 4-manifolds, $\hat{A}(M) = -\frac{1}{8}\mathrm{sig}(M)$ where $\mathrm{sig}(M)$ denotes the signature of M.

For spin manifolds of dimension $n = 4k$, Atiyah and Singer showed that $\hat{A}(M)$ is equal to the topological index of the chiral Dirac operator $\not{D}^+ \colon \mathcal{S}^+(M) \to \mathcal{S}^-(M)$ of the irreducible complex spinor bundle over M. Thus the proof of this corollary uses the index theorem applied to $\not{D}^+$: if M has positive scalar curvature, then by Lichnerowicz's formula (Theorem 2.4.6) it admits no harmonic spinors for any spinor bundle, meaning that $\ker \not{D}^+ = 0$. Thus although $\mathrm{ind}_a(\not{D})$ always vanishes, $\mathrm{ind}_a(\not{D}^+)$ also vanishes in the presence of positive scalar curvature, giving us:

$$\hat{A}(M) = \mathrm{ind}_t(\not{D}^+) = \mathrm{ind}_a(\not{D}^+) = \mathrm{ind}_a(\not{D}) = 0$$

[1]The original proof of the index theorem is in a series of five papers, [AS68b, AS68a, AS68c, AS71a, AS71b]. However, we recommend Chapter III of [LM89] for a comprehensive approach, especially with Dirac operators in mind.

Further developments using the Dirac operator approach use differing extensions of the above technique; commonly they construct a particular graded spinor bundle on some class of manifolds and apply the index theorem to its chiral Dirac operator. Of particular note is an extension of the $\hat{A}$-genus, denoted $\hat{\mathcal{A}}$, which is a graded ring homomorphism from the spin cobordism ring to the real K-theory of a point. Some such results include the following two theorems of Hitchin in [Hit74].

Theorem 2.5.4 (Hitchin)**.** *Let M be a compact spin manifold with positive scalar curvature. Then $\hat{\mathcal{A}}([M]) = 0$, where $[M]$ denotes its spin cobordism class.*

Theorem 2.5.5 (Hitchin)**.** *In every dimension $n \equiv 1$ or $2 \mod 8$, $n > 8$, there exist exotic spheres which do not admit metrics of positive scalar curvature.*

We also have the following results of Gromov and Lawson. In the first theorem, the dimension restriction arises from the use of the h-cobordism theorem.

Theorem 2.5.6 (Gromov-Lawson, [GL80a])**.** *Let M^n be a compact, simply connected manifold with $n \geq 5$.*

i. If M is not spin, then M carries a metric of positive scalar curvature.

ii. If M is spin, and M is spin cobordant to a manifold with positive scalar curvature, then M admits a metric of positive scalar curvature.

Though not directly related to topology, they were also able to prove the following powerful result.

Theorem 2.5.7 (Gromov-Lawson, [GL83])**.** *A compact manifold admitting a metric of nonpositive sectional curvature cannot carry a metric with positive scalar curvature.*

The final result we mention is of particular importance for the search for 3-dimensional MOTS's.

Theorem 2.5.8 (Gromov-Lawson, [GL80b])**.** *No compact 3-manifold containing a $K(\pi, 1)$ in its prime decomposition can carry positive scalar curvature.*

The Dirac operator approach accounts for most of the known topological obstructions to positive scalar curvature. However, there are many questions it leaves open, particularly in low dimensions. The minimal hypersurfaces approach, while generating fewer results, fills in many of these gaps, as we see below.

3 Minimal Hypersurfaces

The idea behind the minimal hypersurfaces approach is that the geometry of a manifold should in some way inform the geometry of its minimal hypersurfaces, and vice versa. This connection was first made in the context of surfaces in 3-manifolds by Schoen-Yau in [SY79a] and Sacks-Uhlenbeck in [SU81]. Because the relationship between geometry and topology is well-known for surfaces, this led to new topological obstructions to metrics of positive scalar curvature for 3-manifolds, as demonstrated in [SY79a].

Almost immediately, this technique was generalised using an inductive family of hypersurfaces, each contained within the previous, in [SY79c]. This generalisation gives a constraint on positive scalar curvature in terms of cohomological data. However, it was not extended beyond dimension 7 for nearly 40 years, as the technique of construction introduces singular sets in the hypersurfaces that grow in dimension with the dimension of the ambient manifold. In a 2017 preprint ([SY17]), Schoen and Yau generalised their approach to all dimensions by working around the singular sets.

We will begin our exploration of these results by first looking at the base case of this induction on dimension: that is, the case of surfaces in 3-manifolds. In this setting, we can get topological obstructions from the fundamental group. We will then take a look at the general inductive argument. We generally omit proofs of the existence of these minimal hypersurfaces, which involve the machinery of geometric measure theory. Instead, we focus on their relationship to positive scalar curvature and their topological implications.

3.1 Minimal Surfaces in 3-manifolds

Let (M, g) be a compact, orientable Riemannian 3-manifold. We take "surface", to mean a compact, orientable surface without boundary. Recall that a *minimal* submanifold is one whose mean curvature vanishes identically, and that minimal surfaces are locally area minimising (see [CM11] for details).

We begin by making two observations. The first one is a simple lemma about surfaces.

Lemma 3.1.1. *Let Σ be a surface of genus $g \geq 1$ and suppose $\pi_1(M)$ has a subgroup H that is isomorphic to $\pi_1(\Sigma)$. Then there exists an immersion $\psi\colon \Sigma \to M$ such that its induced map on fundamental groups ψ_* has $\psi_*(\pi_1(\Sigma)) = H < \pi_1(M)$.*

Proof. Let $p \in M$. Because $\pi_1(\Sigma) \cong H < \pi_1(M) = \pi_1(M, p)$, there exist elements

$a_1, b_1, \ldots, a_g, b_g$ in $\pi_1(M,p)$ that generate:

$$H \cong \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1][a_2, b_2] \ldots [a_g, b_g] \rangle$$

Choose representative based loops $\bar{a}_1, \bar{b}_1, \ldots, \bar{a}_g, \bar{b}_g$ for these elements, and note that the composite loop γ given by the relator in the above presentation must therefore bound a disk D in M. Identify Σ with the $4g$-gon with sides labelled $\bar{a}_1, \bar{b}_1, \ldots, \bar{a}_g, \bar{b}_g$ in the usual way; then ψ sending this $4g$-gon to D with boundary γ along the side identifications given has $\psi_*(\pi_1(\Sigma, \psi^{-1}(p))) = H$. □

Separately, we also have the following theorem, which is presented in Schoen and Yau's original paper, [SY79a].

Theorem 3.1.2 (Schoen-Yau, 1979)**.** *Suppose that M has positive scalar curvature. Then M cannot have any immersed stable minimal surfaces of positive genus.*

Proof. Suppose to the contrary that $\phi\colon \Sigma \to M$ is a stable minimal immersion for Σ a surface of genus $g \geq 1$. Let $\{e_1, e_2, e_3\}$ be a positively oriented local orthonormal frame for TM such that e_1 and e_2 are in $T\Sigma$ and e_3 is the normal unit vector to Σ. Since the codimension of Σ in M is 1, let B_{ij} be such that $B(e_i, e_j) = B_{ij} e_3$. Note also that for these coordinates, $\mathrm{Ric}(e_i, e_3) = K(e_i, e_3)$, the sectional curvature of the plane spanned by e_i and e_3. Here, $\langle \ , \ \rangle$ denotes the metric in M.

Σ being minimal implies that:

$$B_{11} + B_{22} = 0$$

and stability implies that the stability condition is satisfied for any smooth function $f\colon \Sigma \to \mathbb{R}$:

$$\int_\Sigma \left(\sum_{i=1}^{2} \mathrm{Ric}(e_i, e_3) + \sum_{i,j=1}^{2} B_{ij}^2 \right) f^2 \, d\mu \leq \int_\Sigma |\nabla f|^2 \, d\mu$$

Choose $f = 1$. Then the inequality becomes:

$$\int_\Sigma \mathrm{Ric}(e_1, e_3) + \mathrm{Ric}(e_2, e_3) + \sum_{i,j=1}^{2} B_{ij}^2 \, d\mu \leq 0$$

$$\int_\Sigma K(e_1, e_3) + K(e_2, e_3) + \sum_{i,j=1}^{2} B_{ij}^2 \, d\mu \leq 0$$

Separately, we apply the Gauss equation to obtain:

$$\begin{aligned}\langle R^M(e_1, e_2)e_2, e_1\rangle = & \\ & \langle R^\Sigma(e_1, e_2)e_2, e_1\rangle + \langle B(e_1, e_2), B(e_2, e_1)\rangle - \langle B(e_1, e_1), B(e_2, e_2)\rangle \\ =& \langle R^\Sigma(e_1, e_2)e_2, e_1\rangle + B_{12}B_{21} - B_{11}B_{22}\end{aligned}$$

Since e_1 and e_2 are orthonormal, and with $B_{22} = -B_{11}$, this gives:

$$\begin{aligned}K^M(e_1, e_2) &= K^\Sigma(e_1, e_2) + B_{12}^2 + B_{11}^2 \\ &= K^\Sigma(e_1, e_2) + \frac{1}{2}\sum_{i,j=2}^{2} B_{ij}^2\end{aligned}$$

Now above, $K(e_i, e_3)$ were the sectional curvatures in M. The scalar curvature S of M is given by $K^M(e_1, e_2) + K^M(e_1, e_3) + K^M(e_2, e_3)$ by the above noted equivalence between the Ricci and sectional curvatures in this setup, and so we substitute back into our inequality to get:

$$\int_\Sigma S - K^\Sigma(e_1, e_2) + \frac{1}{2}\sum_{i,j=1}^{2} B_{ij}^2 \; d\mu \leq 0$$

$$\int_\Sigma S + \frac{1}{2}\sum_{i,j=1}^{2} B_{ij}^2 \; d\mu - \int_\Sigma K^\Sigma(e_1, e_2) \; d\mu \leq 0$$

But $K^\Sigma(e_1, e_2)$ is just the Gaussian curvature of Σ, so by the Gauss-Bonnet theorem we have

$$\int_\Sigma K^\Sigma(e_1, e_2) \; d\mu \leq 0$$

So since S was assumed to be positive, we have a contradiction. □

We now want to relate these two results by showing that the immersion from Lemma 3.1.1 can be taken to be minimal and stable, hence giving us a fundamental group obstruction to positive scalar curvature. To describe the approach, first recall the following standard definitions and results, as found in [CM11].

Definitions 3.1.3. Let (Σ, h) be a surface and (M, g) be a Riemannian manifold, and following [SY79a] and [SU81], let M be isometrically embedded in $\mathbb{R}^m$ to simplify the target metric. This can be done by the Nash embedding theorem; see [Nas56].

1a. We say that two metrics h and h' on Σ are *conformal* if there exists a nowhere-vanishing smooth function $\lambda\colon \Sigma \to \mathbb{R}$ such that $h' = \lambda^2 h$. This allows us to partition the space of metrics on Σ into *conformal structures*, where h and h' are in the same class if h and h' are conformal.

1b. We say a map $f\colon (\Sigma, h) \to (M, g)$ is *conformal* if there exists such a λ such that $f^*g = \lambda^2 h$.

2. The *energy* of such a map is the integral:

$$
\begin{aligned}
E(f) &= \frac{1}{2}\int_\Sigma \mathrm{tr}_h(f^*g)\ d\mu \\
&= \frac{1}{2}\int_\Sigma |\nabla f|^2\ d\mu
\end{aligned}
$$

where $d\mu$ is induced by h.

3. f is said to be *harmonic* if $\nabla \cdot \nabla f = 0$. Equivalently, if we choose coordinates on Σ, each component of f must have vanishing Laplace-Beltrami operator.

Some standard results relating these definitions are the following:

Proposition 3.1.4. *Let $V \subset \mathbb{R}^2$. Then $x\colon V \to \mathbb{R}^m$ is harmonic if it is a critical point for the energy functional for compactly supported variations.*

This is often given as the definition for harmonic maps in the context of minimal surfaces in $\mathbb{R}^3$. However, it holds in our case for local parametrisations of subsets of Σ. We also have:

Proposition 3.1.5. *The energy of a map $f\colon (\Sigma, h) \to (M, g)$ is constant when varying h within its conformal class.*

Again, for proofs, see [CM11]. We can now state the lemma we use to find minimal surfaces in 3-manifolds.

Lemma 3.1.6. *If $f : \Sigma \to M$ is both conformal and harmonic, then the image of f is a minimal surface.*

Proof. Embed M isometrically in $\mathbb{R}^m$. Let $U \subset \Sigma$ be an open neighbourhood, and let $\varphi\colon U \to V \subset \mathbb{R}^2$ be a chart and pull the metric h back to V via φ^{-1}. Let (u, v) be coordinates on V such that $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ form an orthonormal frame for V with respect to $(\varphi^{-1})^*h$. Then $x\colon V \to \mathbb{R}^m$ given by $x = f \circ \varphi^{-1}$ is a parametrisation of $f(U) \subset \mathbb{R}^n$. Thus it suffices to show that $x(V)$ has vanishing mean curvature in $\mathbb{R}^m$.

That f is conformal means that at each point in V,

$$
\left|\frac{\partial \vec{x}}{\partial u}\right|^2 = \left|\frac{\partial \vec{x}}{\partial v}\right|^2 = \lambda^2 \qquad \text{and} \qquad \left\langle \frac{\partial \vec{x}}{\partial u}, \frac{\partial \vec{x}}{\partial v} \right\rangle = 0
$$

Now as we're working in $\mathbb{R}^m$ with the standard metric, the mean curvature of $\vec{x}(V)$ at $p \in V$ is given by:

$$H_p = \frac{1}{2}\sum_i B(e_i, e_i)$$

for $\{e_1, e_2\}$ an orthonormal basis for $T_{x(p)}x(V) \subset T_{x(p)}\mathbb{R}^m$. Let $e_1 = \frac{1}{\lambda}\frac{\partial \vec{x}}{\partial u}$ and $e_2 = \frac{1}{\lambda}\frac{\partial \vec{x}}{\partial v}$ so that e_1 and e_2 form such a basis (since $\frac{\partial \vec{x}}{\partial u} = x_*(\frac{\partial}{\partial u})$). Then:

$$\begin{aligned}
H_p &= \frac{1}{2}(B(e_1, e_1) + B(e_2, e_2)) \\
&= \frac{1}{2\lambda^2}\left(B\left(\frac{\partial \vec{x}}{\partial u}, \frac{\partial \vec{x}}{\partial u}\right) + B\left(\frac{\partial \vec{x}}{\partial v}, \frac{\partial \vec{x}}{\partial v}\right)\right) \\
&= \frac{1}{2\lambda^2}\left(\nabla^{\mathbb{R}^n}_{\frac{\partial \vec{x}}{\partial u}}\frac{\partial \vec{x}}{\partial u} - \nabla^V_{\frac{\partial}{\partial u}}\frac{\partial}{\partial u} + \nabla^{\mathbb{R}^n}_{\frac{\partial \vec{x}}{\partial v}}\frac{\partial \vec{x}}{\partial v} - \nabla^V_{\frac{\partial}{\partial v}}\frac{\partial}{\partial v}\right) \\
&= \frac{1}{2\lambda^2}\left(\frac{\partial^2 \vec{x}}{\partial u^2} + \frac{\partial^2 \vec{x}}{\partial v^2}\right)
\end{aligned}$$

But f is also harmonic, meaning each component of $\vec{x}$ is harmonic. Hence $\frac{\partial^2 \vec{x}}{\partial u^2} + \frac{\partial^2 \vec{x}}{\partial v^2} = 0$, and so the mean curvature at each $p \in V$ vanishes. □

Thus we can break up the process of finding a minimal surface into two steps. First, we fix a conformal class $[h]$, pick a representative metric h on our surface Σ, and minimise the energy over a suitable family of maps to find a harmonic map ϕ_h with respect to this metric. This minimisation assigns to each conformal structure on Σ an energy, which is the energy of that harmonic map. We then minimise energy again over all conformal classes. The map ϕ_h corresponding to this minimiser will itself be conformal. Hence the resulting map will be both harmonic and conformal, therefore minimal. One can also show that it is area-minimising, thus stable (see Chapter 1, §8 of [CM11]). This solution was obtained independently by Schoen-Yau in [SY79a] and Sacks-Uhlenbeck in [SU81], although it has the same general structure as the argument used by Douglas and Radó in the solution to Plateau's problem (again, see [CM11]). However, there are some deviations from the classical solution in each step, which we now note. Throughout, let (M, g) and Σ be as above, and fix an isometric embedding of M in $\mathbb{R}^m$, with $\langle\ ,\ \rangle$ denoting the standard inner product in $\mathbb{R}^m$ (which then agrees with g).

In the first step, the only deviation from the Douglas-Radó solution is that the maps we minimise over must induce the right injection on fundamental groups. To do this, one shows that even square integrable maps $f\colon \Sigma \to \mathbb{R}^m$ with square integrable derivatives (taken in the sense of distributions) can be taken to be continuous on representatives of generators of $\pi_1(\Sigma, p)$ and disks bounded by them. Thus we can define the induced map on fundamental groups for maps f which are not necessarily continuous. The harmonic map ϕ_h minimising energy, however, ends up smooth. The details of this argument can be found in [SY79a].

The greater deviation is in the second step of the proof, in which we vary the conformal class $[h]$. In the Douglas-Radó solution, the candidate surface is given a conformal parametrisation, and then a corresponding harmonic map is found. Here, however, we cannot reparametrise Σ without changing its metric, and so we must vary the conformal class of all of Σ at once. However, the space of conformal structures on Σ is quite unwieldy and has a lot of symmetries, and so we introduce different structures to minimise over.

Definition 3.1.7. Let Σ be a surface of genus g and Σ_h denote Σ with the metric h. The corresponding *Teichmüller space* is the space of pairs (Σ_h, τ) with $\tau \colon \Sigma \to \Sigma_h$ a homeomorphism such that $(\Sigma_h, \tau) \sim (\Sigma_{h'}, \tau')$ if $\tau' \circ \tau^{-1}$ is isotopic to a conformal map.

The *moduli space* M_g of Σ is the quotient of T_g by the action of its mapping class group $\mathrm{MCG}(\Sigma)$ given, for $\sigma \in \mathrm{MCG}(\Sigma)$, by:

$$\sigma \colon (\Sigma_h, \tau) \mapsto (\Sigma_h, \tau\sigma^{-1})$$

Both T_g and M_g, are important objects of study in a variety of subfields within geometry and topology. For $g \geq 2$, T_g is naturally homeomorphic to a ball of dimension $6g - 6$, and M_g has an orbifold structure. Thus passing the minimisation problem to $\bar{E} \colon T_g \to \mathbb{R}$ given by $\bar{E}(\Sigma_h, \tau) = E(\phi_h)$ allows us to obtain a global minimum for energy on T_g.

As all the ϕ_h are harmonic, the map Φ_h corresponding to this global minimum is then both harmonic and conformal, and one can show that it is area minimising among all ϕ_h. This gives rise to the following theorem, as in [SU81].

Theorem 3.1.8. *If $(\Sigma_h, \tau) \in T_g$ is an absolute minimum for the energy function $E \colon T_g \to \mathbb{R}$, then the associated map $\phi_h \colon (\Sigma, h) \to M$ is a branched minimal immersion.*

Thus after minimising over Teichmüller space, we only need to worry about the possible branch points. Luckily, we have the following theorem.

Theorem 3.1.9 (Gulliver-Osserman)**.** *A locally area minimising surface has no interior branch points.*

This follows from results in [Gul73, Oss70]. Thus we obtain the following:

Theorem 3.1.10 (Schoen-Yau, [SY79a]). *If $\pi_1(M)$ contains a subgroup isomorphic to a surface group, then M cannot carry positive scalar curvature.*

This is a particularly nice result for 3-manifold topologists. It also precedes the work of Gromov-Lawson on $K(\pi, 1)$'s, and techniques from its proof formed the foundation for Schoen and Yau's later proof in [SY79b] of the positive mass theorem in general relativity.

3.2 General Hypersurfaces

This section briefly summarises the results of [SY79c] and [SY17], which generalises the above techniques to higher dimensions. We begin with the basic definition:

Definition 3.2.1. Let $M^n = \Sigma_n$ be a compact, oriented Riemannian manifold. A *minimal k-slicing* of M is a family of nested hypersurfaces $\Sigma_k \subset \Sigma_{k+1} \subset \cdots \subset \Sigma_n$ such that each Σ_j minimises a particular weighted volume functional $V_{\rho_{j-1}}$.

Schoen and Yau then prove that if Σ_j admits a metric of positive scalar curvature, then Σ_{j-1} does as well. The proof follows a similar structure to that of Theorem 3.1.2, in which a stability condition coming from the volume minimisation property is manipulated to create a bound on scalar curvature.

The functional V_{ρ_j} is constructed inductively, with a slight modification if dim $M \geq 8$. For $V_{\rho_{n-1}}$, choose an oriented, volume minimising hypersurface $\Sigma_{n-1} \subset M$ and let S_{n-1} denote its second variation form for volume. Because Σ_{n-1} is volume minimising and thus stable, S_{n-1} must have its first eigenvalue nonnegative. We then choose a corresponding eigenfunction u_{n-1} and use it as a weight ρ_{n-1} for the volume functional on submanifolds Υ_{n-2} with induced measure from the Riemannian metric $d\mu_{n-2}$ of Σ_{n-1} so that:

$$V_{\rho_{n-1}}(\Upsilon) = \int_{\Upsilon} \rho_{n-1} \, d\mu_{n-2}$$

We then find Σ_{n-2} a hypersurface of Σ_{n-1} minimising $V_{\rho_{n-1}}$. Given such a hypersurface, we repeat the above process by letting S_{n-2} be the second variation form for Σ_{n-2} (with respect to $V_{\rho_{n-1}}$), noting that it has a nonnegative first eigenvalue and hence choosing a first positive eigenfunction u_{n-2}, and setting $\rho_{n-2} = u_{n-2}\rho_{n-1}$. The above process produces the required result if $\dim M \leq 7$, as shown in [SY79c], but if $\dim M > 7$, then the constructed hypersurfaces may contain singular sets; we address this case in a moment.

The connection to topology comes from the following theorem.

Theorem 3.2.2. *Let M^n be a compact, oriented manifold, $n \leq 7$, and let $1 \leq k \leq n-1$. Suppose there exist $\alpha_1, \ldots, \alpha_k \in H^1(M; \mathbb{Z})$ such that $\alpha_{n-k} \frown \cdots \frown \alpha_1 \frown [M] \neq 0$. Then there exists a minimal k-slicing with Σ_j a representative for the homology class $\alpha_{n-j} \frown \cdots \frown \alpha_1 \frown [M]$.*

Putting this all together, we obtain our desired topological obstruction.

Theorem 3.2.3. *Let M^n be a compact, oriented manifold, $n \leq 7$, with a metric of positive scalar curvature. If $\alpha_1, \ldots, \alpha_{n-2}$ are classes in $H^1(M; \mathbb{Z})$ such that $\sigma \in H_2(M; \mathbb{Z})$ given by $\sigma = \alpha_{n-2} \frown \cdots \frown \alpha_1 \frown [M]$ is nonzero, then σ can be represented by a sum of spheres. If α_{n-1} is any other class in $H^1(M; \mathbb{Z})$, then $\alpha_{n-1} \frown \sigma = 0$.*

In particular, if M has classes $\alpha_1, \ldots, \alpha_{n-2} \in H^1(M; \mathbb{Z})$ such that $\alpha_1 \frown \cdots \frown \alpha_{n-2} \frown [M] \neq 0$, then M cannot carry a metric of positive scalar curvature.

Proof. If M has such classes $\alpha_1, \ldots, \alpha_{n-2} \in H^1(M; \mathbb{Z})$, then by Theorem 3.2.2, there exists a minimal 2-slicing of M. If M has positive scalar curvature, then the 2-slice Σ_2 must admit a metric of positive scalar curvature. Hence by the Gauss-Bonnet theorem, we have that each component of Σ_2 is topologically spherical. Now if α_{n-1} is any other nonzero class in $H^1(M; \mathbb{Z})$, then we can pull it back via the inclusion $i \colon \Sigma_2 \to M$ to $i^*\alpha_{n-1} \in H^1(\Sigma_2; \mathbb{Z})$, where it must vanish because $H^1(\Sigma_2; \mathbb{Z}) = 0$. Thus $\alpha_{n-1} \frown [\Sigma_2] = 0$ in $H_1(\Sigma_2; \mathbb{Z})$, and so $\alpha_{n-1} \frown \sigma = 0$ in $H_1(M; \mathbb{Z})$. The remaining statement follows immediately. □

In [SY17], Schoen and Yau claim to extend this result by removing the dimension restriction $n \leq 7$. To allow for the possibility of singular sets in each slice, Schoen and Yau alter their construction of V_{ρ_j} by using a modified form of the second variation form, which allows for closer control on the size of the singular sets of each Σ_j. With enough control at each stage, they construct minimisers of the appropriate volume functional and relate them to $H_j(M, \mathcal{S}_j; \mathbb{Z})$ where $\mathcal{S}_j$ is the singular set of Σ_j. This result has not been published yet, but if true, it provides a powerful topological restriction to positive scalar curvature. An immediate corollary would be that the n-torus does not admit a metric of positive scalar curvature for any n, and neither does any manifold obtained by taking a connect sum with the n-torus (these results hold for $n \leq 7$). In general, its implication is that if a manifold has "enough" intersecting hypersurfaces, each nontrivial in homology, then it cannot carry a metric of positive scalar curvature.

Final Remarks

Apart from some results based on these techniques that we did not have space to mention and more recent work in Seiberg-Witten theory for 4-manifolds, the above represents a large part of what is known about topological obstructions to positive scalar curvature. Of course, remembering our original question, we note that it is unlikely that all manifolds admitting metrics of positive scalar curvature can be realised as black hole horizons; Theorem 1.2.3 does not guarantee the existence of such a spacetime. However, a family of solutions to the 5-dimensional Einstein equations has been found realising $S^1 \times S^2$ as a black hole horizon. These "black ring" solutions, and other higher-dimensional black hole solutions in general, are of interest both in their own right and in certain physical theories; for more information, see [Hor12]. The topological restrictions we have given here thus provide, if not a roadmap, then a map of blocked roadways in the search for more of these solutions in higher dimensions.

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