Concordance of surfaces in 4-manifolds

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Abstract Concordance of classical knots is an important and widely studied topic in low-dimensional topology. However, the theory of concordance for knotted surfaces in 4-manifolds differs dramatically from the lower-dimensional case. In this chapter, we aim to outline some of the foundational results in surface concordance, with a particular focus on the relevant techniques for proving obstructions.

1 Concordance in simply connected 4-manifolds

1.1 Preliminaries

We recall the definition of concordance for two submanifolds of an ambient manifold.

Definition 1 $A, B \subset X$ are *concordant* if there is an embedding $M \times I \to X \times I$ sending $M \times \{0\}$ to $A \times \{0\}$ and $M \times \{1\}$ to $-B \times \{1\}$.

The goal of this section will be to prove the following theorem.

Theorem 1 (Kervaire 1965) *Every embedding of S^2 into S^4 bounds an embedded copy of B^3 into B^5.*

Although Kervaire's original proof ([6]) was written in the smooth setting, later work of Quinn ([10]) has shown that the same proof technique can be extended to work in the topological setting; that is, for locally flat embeddings of S^2 and B^3 into S^4 and B^5.

Before beginning the proof, we start by discussing spin structures, as they will play a critical role in the argument. The following is not the standard definition of spin structure, but it will serve for our purposes.

Definition 2 Let M be a smooth n-manifold with $n \geq 3$. A *spin structure* on M is a trivialization of its tangent bundle TM on the 1-skeleton of M such that this

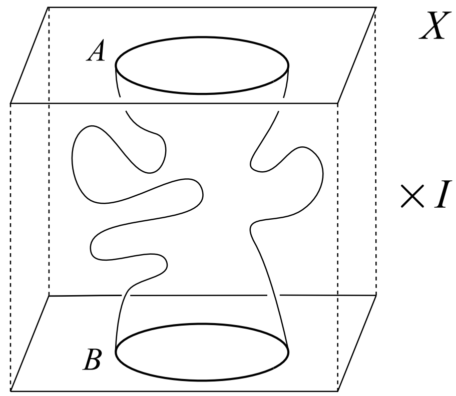

Fig. 1 Schematic of a concordance.

trivialization can be extended over its 2-skeleton, taken up to homotopy. M is said to be *spin* if such a trivialization exists.

One can think of spin structures as being analogous to orientations in the following sense. Recall that an orientation on M can be defined as a choice of trivialization of the 0-skeleton of M that extends over the 1-skeleton. M is orientable exactly when the first Stiefel-Whitney class of its tangent bundle vanishes. Similarly, M is spin exactly when the second Stiefel-Whitney class of TM also vanishes.

Of particular importance is the following situation. Suppose γ is a loop in M^n. Then a homotopy class of trivializations of TM restricted to γ can be identified with an element of $\pi_1(\mathrm{SO}(n))$. For $n \geq 3$, this group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. So there are exactly two distinct spin structures over γ. However, if γ bounds a disk D in M, then as D is contractible, $TM|_D$ has a unique trivialization. Thus for each null-homotopic loop γ in M, one spin structure over γ will extend over a disk D that it bounds, and the other will not.

More perspectives on spin structures can be found in Section 5.6 of [3] or in Chapter IV of [7].

The proof of Kervaire's theorem will also make use of the following theorems regarding spin 3-manifolds. Proofs of all of these theorems can be found in Chapter VII of [7].

Theorem 2 (Stiefel 1963, [14]) *All (orientable) 3-manifolds are spin.*

Theorem 3 (Milnor 1963, [9]) *Suppose M and N are 3-manifolds with spin structures s_M and s_N, respectively. Then there exists a spin 4-manifold with boundary X with spin structure s_X such that $\partial X = M \sqcup -N$ and s_X restricts to s_M and s_N on the boundary.*

Theorem 4 (Kaplan 1979, [5]) *Suppose M, N, and X are as in the previous theorem along with their spin structures. Then X can be chosen such that it is obtained from $M \times [0, 1]$ by adding only 2-handles.*

Remark 1 As an application of Kaplan's theorem, we obtain a strengthening of the Lickorish-Wallace theorem to the statement that all 3-manifolds are obtained from S^3 via surgery on a link where the surgery coefficients are all even integers.

1.2 Proof of main theorem

We now turn to the proof of Kervaire's theorem. This is not quite his original proof, and is adapted from an extension of Sunukjian ([16]), which we discuss after the proof.

Proof. Let $S \cong S^2$ be embedded in S^4. Since $H_2(S^4;\mathbb{Z}) \cong 0$, S bounds a smoothly embedded 3-manifold Y into S^4. We can then push the interior of Y into B^5 to obtain an embedding that will serve as a starting point for our construction.

Since all 3-manifolds are related by Dehn surgery, there exists some surgery we can perform on Y to turn it into B^3. However, in order to preserve the embedding into B^5, we need to perform this surgery *ambiently*; that is, we want to achieve Dehn surgery along a desired curve in Y by identifying it as the attaching sphere of some 4-dimensional 2-handle embedded in B^5 and then replacing a neighborhood of the curve in Y with a thickening of the core of the 2-handle.

To do this, we will need three things to be true:

1. We need whatever curves we need to surger Y along to get B^3 to bound disks into $B^5 \setminus Y$ which can serve as the cores of our 2-handles.
2. We need to be able to find a 2-dimensional thickening of each of these disks in B^5 such that the restriction of these disk bundles to their boundary curves agree with the normal bundle of these curves in Y.
3. We need to be able to control the framing of the Dehn surgery we perform.

To address (1), we need to be careful about what we mean by "pushing" Y into B^5. Fix coordinates on B^5 and a self-indexing Morse function f on Y such that $f^{-1}(0) = S$. A standard argument in Morse theory shows that we can arrange f such that it has only one local maximum. We can then isotope Y in B^5 by sending each $x \in Y$ "straight down" into B^5 such that the distance from its image to S^4 is precisely $f(x)$ with respect to radial height in B^5, as shown in Figure 2. Note that with respect to a radial height Morse function $\bar{f}$ on B^5, Y now has a single minimum.

We now apply a "rising waters" argument (see Prop. 6.2.1 of [3]) to obtain a handle decomposition of $B^5 \setminus Y$. Since Y has a single minimum with respect to $\bar{f}$, it has a single 0-handle in the handle decomposition induced by $\bar{f}|_Y$. Hence its complement in B^5 has a single 1-handle in its decomposition induced by $\bar{f}$. From this we conclude that $\pi_1(B^5 \setminus Y)$ is cyclic and is generated by a meridian of Y. We can in fact deduce that $\pi_1(B^5 \setminus Y)$ is isomorphic to $\mathbb{Z}$ by Alexander duality.

Now suppose c is a curve in Y along which we would like to do Dehn surgery. To consider a homotopy class of c in $\pi_1(B^5 \setminus Y)$, we need to choose a pushoff of c into a tubular neighborhood $N_{B^5}(Y)|_c$ of Y in B^5. This neighborhood is diffeomorphic to a 2-dimensional disk bundle over c, and thus the isotopy classes of such pushoffs

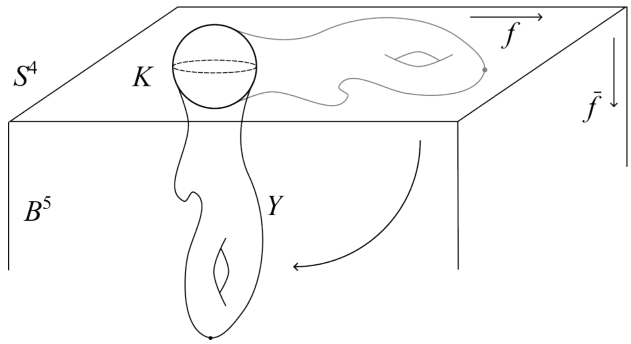

Fig. 2 Isotopy of Y to Morse position in B^5.

are in bijection with $\pi_1(\mathrm{SO}(2)) \cong \mathbb{Z}$. However, since $\pi_1(B^5 \setminus Y)$ is generated by a meridian of Y, these pushoffs correspond exactly to elements of $\pi_1(B^5 \setminus Y)$, and so there is a unique null-homotopic pushoff that corresponds to the trivial element. Thus c bounds some disk D into the complement of Y in B^5. Note that we can take this disk to be smoothly embedded, as its codimension in B^5 is sufficiently high to resolve any double points.

We now proceed to (2). Recall that our goal is to find a 2-disk bundle over the disk D embedded in B^5 that agrees with the normal bundle of c in Y on its boundary. This is equivalent to finding a trivialization of the normal bundle $N_Y(c)$ of c in Y that extends over D.

Now trivializations of $N_Y(c)$ up to homotopy are in bijection with $\pi_1(\mathrm{SO}(2)) \cong \mathbb{Z}$. (These trivializations are precisely the $\mathbb{Z}$ framings one can associate to a knot in a 3-manifold). However, trivializations of $N_{B^5}(D)|_c$ are in bijection with $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2\mathbb{Z}$. These two trivializations correspond to the two possible parities of trivializations of $N_Y(c)$. Since D is contractible, there is a unique trivialization of $N_{B^5}(D)$. Thus we expect that a trivialization of $N_Y(c)$ will extend over D if and only if it has the same parity as the unique trivialization of $N_Y(D)$ when the latter is restricted to c.

This argument shows that, given c and a framing k, a suitable 4-dimensional 2-handle with attaching sphere c and framing k in B^5 can be found when the framing is of a particular parity determined by the core disk D. Thus we can ambiently surger Y along c, but we are restricted in our choice of surgery coefficient. This brings us to (3).

So far, we know that we can perform some ambient surgeries on c, but we only know that surgeries of one parity are possible and the surgeries of the other are not. To pin down which are possible, note that B^5 is spin and has a unique spin structure. Thus there is a "preferred" trivialization of TB^5 on its 1-skeleton that extends over the entire 2-skeleton. Restricting this trivialization to c, we get a specific trivialization of $TB^5|_c$ that extends over D. We can then restrict this trivialization to one of $N_Y(c)$ where, because it was induced from a spin structure, it must extend over D.

So, since the trivializations that agree with the spin structure on B^5 extend over D, we can apply Kaplan's theorem (Theorem 4) to see that we can obtain B^3 by such surgeries. □

Note that this proof was not very specific to the case of a sphere. We may just has well have started with some other knotted surface and used this technique to construct a handlebody it bounds. Additionally, the only fact about S^4 that we used was that it is simply connected and spin. These observations, together with some work in the non-spin case, led to the following generalization.

Theorem 5 (Sunukjian 2015, [16]) *Any two homologous, genus g surfaces in a simply connected 4-manifold X are concordant.*

2 Concordance obstructions

After showing that the theory of concordance is trivial in simply-connected 4-manifolds, we now look for a homotopic pair of surfaces in *some* 4-manifold that are not concordant. We will define an obstruction to concordance of a pair of 2-spheres in any 4-manifold – this will end up being the Freedman-Quinn invariant.

2.1 The Freedman-Quinn invariant

To motivate the definition, we start with a pair of homotopic 2-spheres S_0, S_1 in a 4-manifold X and try to find a concordance between them. We will first appeal to the following theorem from work in [13] and [4].

Theorem 6 (Hirsch, Smale) *Homotopic embedded surfaces in a 4-manifold are regularly homotopic.*

We can think of this regular homotopy of 2-spheres as an immersion of $S^2 \times I$ into $X \times I$, with boundary $S_0 \times \{0\} \sqcup -S_1 \times \{1\}$.

The image of $S^2 \times I$ in $X \times I$ intersects itself in finitely many circles. Since our goal is to find a concordance, we would like to ambiently surger away these self-intersections to obtain an embedding.

View each of these circles c as an element $[c]$ of $\pi_1(X \times I) \cong \pi_1(X)$ by choosing an arc from the basepoint x to c. The preimage of c in $S^2 \times I$ is either two circles (in which case we call c Type I) or one circle covering c twice (in which case c is called Type II).

If c is Type I, then $[c] = 1 \in \pi_1(X)$. If c is Type II, then $[c]^2 = 1 \in \pi_1(X)$. In the latter case, the presence of 2-torsion in $\pi_1(X)$ precludes finding a disk bounded by c in $X \times I$ along which to surger and obtain a concordance. (Note that we do not explain here how to remove nullhomotopic self-intersection curves, just that any issues in doing so do not come from π_1. For details, see [16]).

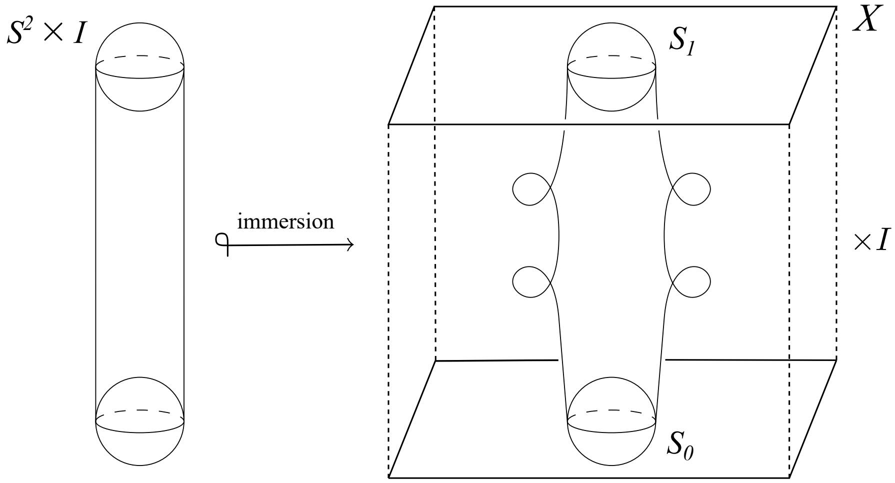

Fig. 3 A regular homotopy between a pair of spheres.

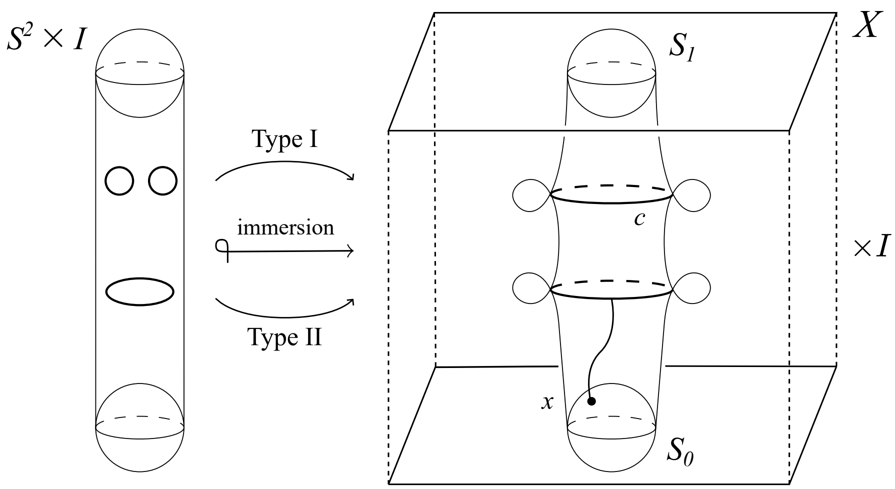

Fig. 4 Self-intersection circles of a regular homotopy between S_0 and S_1.

We have found our obstruction. However, we still need to account for the possibility that a different choice of immersed $S^2 \times I$ could be a concordance. We can compare two such immersions by requiring them to be *based*, i.e. for each $S^2 \times I$ to contain $\{x\} \times I$. By removing a small neighborhood of $\{x\} \times I$ from each $S^2 \times I$ and gluing together what remains, we obtain an immersed S^3 in $X \times I$, which we can view as an element of $\pi_3(X) \cong \pi_3(X \times I)$ measuring the "difference" between the two immersed $S^2 \times I$.

For each $f \in \pi_3(X)$, choose a transversely immersed representative. Define $\mu_3 : \pi_3(X) \to \mathbb{F}_2[\pi_1(X)]$, where $\mathbb{F}_2[\pi_1(X)]$ denotes the group ring, by

$$\mu_3(f) := \sum_{\substack{[c]\in\pi_1(X)\\ [c]^2=1,\ [c]\neq 1}} n_c \cdot [c] \tag{1}$$

where n_c is the number (mod 2) of circles of self-intersection representing $c \in \pi_1(X)$, and $[c]$ is viewed as an element of $\pi_1(X)$ by choosing an arc to the basepoint.

Now we can define the Freedman-Quinn invariant. The invariant was originally defined in Chapter 10, Section 9 of [1], and most of the results in the remainder of this subsection were originally proven in that text. However, those interested are also referred to Schneiderman and Teichner's exposition of the invariant in [11], the language of which more closely matches that of this chapter. It also contains some computational results, noted below.

Definition 3 (Freedman-Quinn invariant) Let S_0 and S_1 be based-homotopic 2-spheres in X^4. Choose a based immersion $h : S^2 \times I \to X \times I$ connecting them. Then

$$\mathrm{fq}(S_0, S_1) := \sum_{\substack{g\in\pi_1(X)\\ g^2=1,\ g\neq 1}} n_g \cdot g \quad \in \mathbb{F}_2[\pi_1(X)]/\mu_3\pi_3(X) \tag{2}$$

where n_g is the number (mod 2) of circles of self-intersection representing $g \in \pi_1(X)$, and $\mu_3\pi_3(X)$ is the image of $\pi_3(X)$ under μ_3.

Remark 2 In the above two definitions, n_c and n_g are taken mod 2, as it is sometimes possible to replace two Type II curves representing the same element of $\pi_1(X)$ with Type I curves.

As an illustrative example, consider the case where S_0 and S_1 are based-concordant. If h is itself a based concordance, it will have no self-intersections, and $\mathrm{fq}(S_0, S_1)$ will be 0. If not, then every self-intersection of h will appear in $\mu_3\pi_3(X)$, as the only self-intersections in the "difference" of h and the concordance will be those in h itself. Then, any contribution to fq vanishes when the quotient is taken, and $\mathrm{fq}(S_0, S_1)$ is 0 as desired. As a consequence, we have the following result:

Corollary 1 *If $fq(S_0, S_1) \neq 0$, S_0 and S_1 are not based-concordant.*

The following theorem is useful in computing the Freedman-Quinn invariant.

Theorem 7 (Schneiderman-Teichner 2022) $\mu_3\pi_3(X) = 1$ *if* $\pi_1(X) = \mathbb{Z}/2\mathbb{Z}$.

In order to loosen the "based" condition in our definition, we will consider the case where there is an *algebraically dual sphere*, or a 2-sphere immersed in X intersecting each S_i in an odd number of points.

Suppose there is a concordance (or homotopy) C between S_0 and S_1, and a dual sphere G in X. Then, in $X \times I$, the intersection of $G \times I$ and C will be a union of circles and arcs with endpoints on $S_0 \times \{0\} \cup S_1 \times \{1\}$.

Since each $G \cap S_i$ contains an odd number of points, at least one arc in $G \times I \cap C$ will have one endpoint on $S_0 \times \{0\}$ and the other on $S_1 \times \{1\}$. Call this arc γ. As $\pi_1 G$ is trivial, γ can be "straightened" inside $G \times I$ so that its intersection with each slice

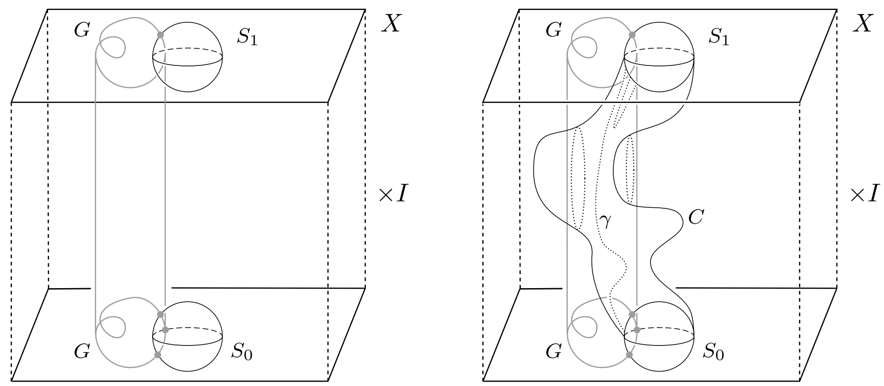

Fig. 5 Left: $G \times I$ in $X \times I$, intersecting each S_i in an odd number of points.
Right: A concordance C between S_0 and S_1, and the intersection $C \cap G \times I$.

$X \times \{t\}$ is a single point. Now γ acts as our $\{\text{basepoint}\} \times I$. Thus, any concordance (or homotopy, by the same argument) will be isotopic to a based one, via a small isotopy preserving the group elements associated to the Type II circles.

We can now define the Freedman-Quinn invariant when there is an algebraically dual sphere, without requiring a based concordance, and obtain the following more general result.

Corollary 2 *If $fq(S_0, S_1) \neq 0$, S_0 and S_1 are not concordant.*

2.2 An example of non-concordant spheres

We now construct a sphere S_0 in some 4-manifold X and a homotopy H from S_0 to some other sphere S_1 such that $\mathrm{fq}(S_0, S_1) \neq 0$. This construction is due to Schwartz in [12].

In order to exploit Theorem 7 we will build X with $\pi_1(X) = \mathbb{Z}/2\mathbb{Z}$ as follows.

1. Start with a single 1-handle attached to B^4, viewed as $S^1 \times D^2 \times I$.
2. Attach 2-handles (with any framing) along the attaching curve γ in each of the two $S^1 \times D^2$-boundary components, as in Figure 6.
3. Attach one more 0-framed 2-handle along a meridian of an attaching curve. This will provide the dual sphere G, intersecting S_0 in one point.

Now, the union of $\gamma \times I$ and the cores of the 2-handles attached along $\gamma \times \{0\}$ and $\gamma \times \{1\}$ is a knotted sphere in X. This will be our S_0. Next, we perform the regular homotopy in Figure 7 to obtain S_1.

The homotopy has a single circle of self-intersection passing once around the core of the solid torus, so $\mathrm{fq}(S_0, S_1) = 1 \cdot g \neq 0$, where g is a generator of $\mathbb{Z}/2\mathbb{Z}$. Thus, S_0 and S_1 are not concordant.

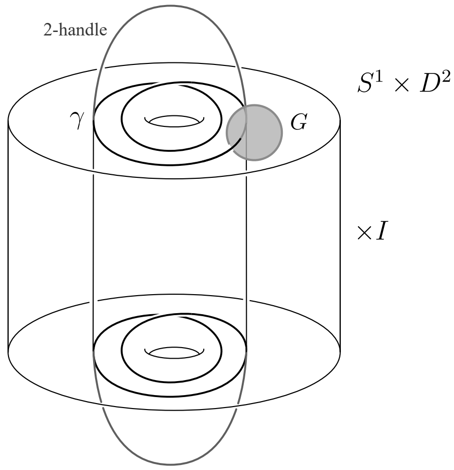

Fig. 6 The construction of X.

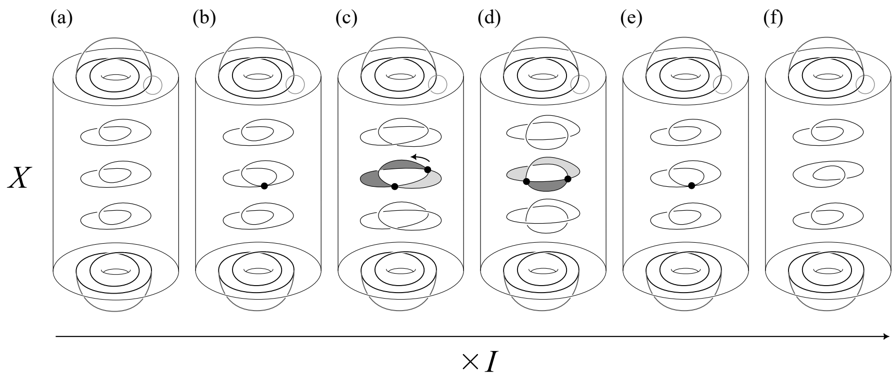

Fig. 7 Read left to right, the figure shows a homotopy between S_0 and S_1 in X. The marked intersection points trace out the circle of self-intersection.
(a) S_0 inside X
(b)-(c) Perform a finger move along the light shaded region.
(c)-(d) Translate one point of self-intersection around the solid torus
(d)-(e) Use the dark shaded region to perform a Whitney move and remove the two points of intersection.
(f) S_1 inside X. The $\gamma \times I$ portion of S_1 performs two full twists in the S^1 direction of $S^1 \times D^2$.

Remark 3 Schwartz's original paper is not written in terms of the Freedman-Quinn invariant – she later includes an appendix explaining her argument in terms of fq.

We conclude with a final application of the Freedman-Quinn invariant to isotopy of spheres.

Theorem 8 (Light Bulb Theorem, Gabai 2017, Schneiderman-Teichner 2022) *If R_0, R_1 are homotopic 2-spheres in a 4-manifold X and there is a sphere G embedded* *in X with trivial normal bundle and $G \cap R_i$ a single point, then R_0, R_1 are isotopic if and only if $fq(R_0, R_1) = 0$.*

Gabai's original paper [2] did not mention the Freedman-Quinn invariant, instead restricting to the case where $\pi_1(X)$ had no 2-torsion. Schneiderman and Teichner [11] then clarified that the Freedman-Quinn invariant was the obstruction in the 2-torsion case, after Schwartz's example [12] demonstrated that there was indeed an obstruction in this setting.

2.3 Framed duals, unframed duals, and the Kervaire-Milnor invariant

Having seen that the Freedman-Quinn invariant can obstruct two 2-knots from being concordant, we want to investigate when the vanishing of fq is enough to conclude that a pair of 2-knots are indeed concordant in a general 4-manifold. Recall that to even define the Freedman-Quinn invariant, it was necessary that the two spheres S_0 and S_1 had an algebraically dual sphere. The following theorem, originally stated in [1] and refined in [8], tells us that if we further require the dual sphere to be *framed*, or have a particular trivialization of its normal bundle, and *geometrically dual*, or intersect in exactly one point, then the Freedman-Quinn invariant is enough to detect concordance.

Theorem 9 (Freedman-Quinn 1990, c.f. Klug-Miller 2019) *Let S_0, S_1 be embedded, homotopic 2-spheres in X^4. Suppose there exists $G \cong S^2$ immersed in X such that G intersects S_0 transversely in a single point and $G \cdot G \equiv 0 \mod 2$. Then S_0 and S_1 are concordant if and only if $fq(S_0, S_1) = 0$.*

The notation $F_1 \cdot F_2$ refers to the geometric intersection number of F_1 and F_2 in X. Thus, requiring $G \cdot G \equiv 0 \mod 2$ means G has even Euler number. We will not go through a full proof of this theorem, but rather focus the importance of this framing condition on G.

Firstly, note that in the presence of a framed geometric dual, any meridian of S_0 is null homotopic in $X \backslash S_0$. We can see this by tubing a meridian to G to get an immersed disk that it bounds.

To illustrate why we need $G \cdot G \equiv 0 \mod 2$, consider the following argument. Suppose S_0, S_1 are homotopic spheres in X^4 and that we want to build a concordance between them. Since they are homotopic, S_0 and S_1 cobound an immersed copy of $S^2 \times I$ in $X \times [0, 1]$. To turn this into a concordance, we first ambiently surger $S^2 \times I$ until it is embedded, at the cost of changing its topology to some other 3-manifold Y. If we can achieve this, then we can proceed as in Kervaire's theorem to surger away that topology without breaking the embedding until we have $S^2 \times I$ again. Suppose for this discussion that we have already done the first step and now want to surger along some curve c to turn Y back into $S^2 \times I$.

In this setup, c will not always bound a disk D into the complement of Y. The theorem is proved by finding an equivariant set of disks in the universal cover of

$X \times I$, but we will ignore this step and instead assume that we found a suitable disk in $(X \times I) \setminus Y$.

Now assuming we have found such a disk D, we again run into a framing problem: of the $\mathbb{Z}$ possible trivializations of $N_Y(c)$, only those of the proper parity will extend over D. However unlike in Kervaire's theorem, we cannot control what parity of Dehn surgery we need to perform on Y to change it into $S^2 \times I$, since we make no assumptions about the ambient manifold. We therefore try to adapt D to some other disk D' via homotopy so that the framing does extend. The homotopy we want to perform is illustrated below. We denote a trivialization of $N_Y c$ by a triple of linearly independent vector fields $\langle v_1, v_2, v_3 \rangle$.

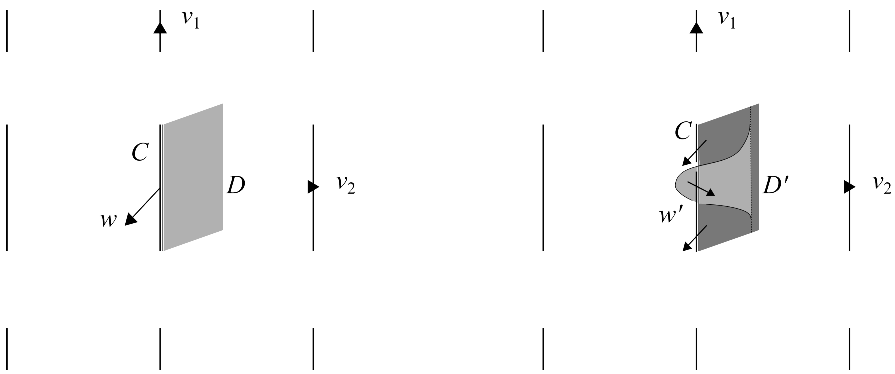

Fig. 8 The disks D and D'. Each grid of lines represents Y, while D is understood to be normal to Y in ambient 5-space. The arrows v_1, v_2, and w indicate normal vector fields to c, together forming a trivialization of $N_Y c$. D' agrees with D except in the neighborhood shown, where it makes an extra twist around c. Thus its normal vector field w' also twists once around c, at the cost of intersecting Y.

Note that D' as illustrated has an extra "twist" in the direction that formerly was normal to Y along c. As a result, since $N_Y D'|_c \cong N_Y c \oplus \langle w' \rangle$, we see that if the trivialization $\langle v_1, v_2, w \rangle$ did not extend over D (where v_1, v_2, w, and w' are as pictured) then $\langle v_1, v_2, w' \rangle$ does extend over D'. However, D' now intersects Y in its interior, so we can't do surgery on it as is.

To fix this, we will need our dual sphere G. Since G intersects S_0 transversely in a single point with $G \cdot G \equiv 0 \mod 2$, tubing D' to a parallel copy of G will allow us to obtain yet another disk D'', now that is disjoint from Y except at its boundary. Crucially, the fact that $G \cdot G \equiv 0 \mod 2$ means that we can extend the trivialization $\langle v_1, v_2, w' \rangle$ over the parallel copy of G to get a trivialization on D'' of the same parity as that of D'.

Thus the existence of a framed geometric dual is crucial to the proof of Theorem 9. But what can we say if there does not exist a framed geometric dual to our spheres? To aid in this discussion, we make the following definition.

Definition 4 An embedded 2-sphere F in a 4-manifold X is s-characteristic if for any immersed 2-sphere G in X, we have that $F \cdot G = G \cdot G \mod 2$.

Note in particular that if S_0 has a framed geometric dual, then S_0 cannot be s-characteristic, but this is not a sufficient condition. However, as shown in the following proposition, if S_0 is not s-characteristic and has *any* geometric dual, we can always find a framed one.

Proposition 1 *Let F be an embedded 2-sphere and G an immersed 2-sphere in X^4. If F intersects G transversely in a single point and F is not s-characteristic, then there exists some immersed 2-sphere G' in X such that F intersects G' transversely in a single point and G' has $G' \cdot G' \equiv 0 \mod 2$.*

Proof. Assume F is not s-characteristic. If $G \cdot G \equiv 0 \mod 2$ then we are done, so assume $G \cdot G \equiv 1 \mod 2$. Since F is not s-characteristic, there exists some 2-sphere S immersed in X such that $F \cdot S \not\equiv S \cdot S \mod 2$. We can then build G' by starting with S and then tubing to parallel copies of G a number of times equal to $F \cdot S - 1$ near all but one of the intersection points, so that the resulting sphere G' intersects F in a single point. Furthermore, $G' \cdot G' = S \cdot S + F \cdot S - 1 \equiv 0 \mod 2$.

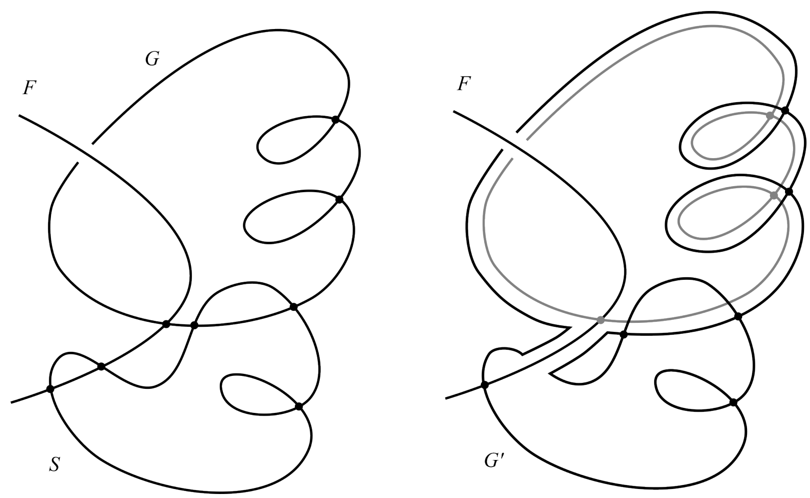

Fig. 9 Tubing S to a parallel copy of G to remove intersection points with F.

□

We conclude this section by defining another invariant that obstructs concordance in the case where S_0 and S_1 are s-characteristic and where they have no framed geometric duals. Consider again our immersed copy of $S^2 \times I$ before any surgery and note the following definition. It is given in the generality of any simply-connected 3-manifold Y.

Definition 5 Let Y be a simply-connected 3-manifold immersed in a 5-manifold Z. For every self-intersection curve $c \subset Y$, choose a path in the image of Y from the basepoint to c and then another, still in the image of Y, such that the path returns to the same point in c but on the other sheet of Y. The corresponding element g_c in $\pi_1 Z$, well-defined up to choice of orientation, is called the sheet change element corresponding to c.

To motivate this definition, consider Figure 10. Since c is double covered by the immersion of Y in Z, the sheet change element g_c passes through the same point of self-intersection in c twice, one on each sheet.

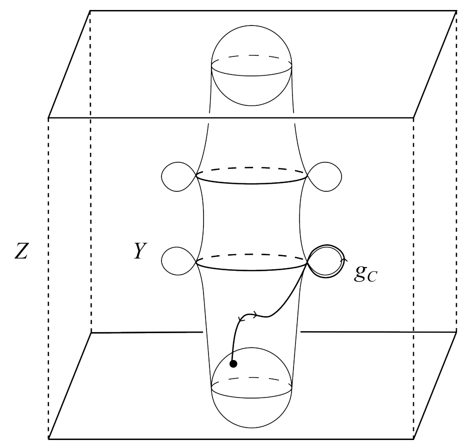

Fig. 10 The sheet change element g_c.

From the sheet change element we can extract the following concordance obstruction. Let H be an immersed copy of $S^2 \times I$ inside $X \times I$ for X a 4-manifold. Let L be the link in $S^2 \times I$ covering the set of self-intersection circles in H. For Type I pairs of preimage circles in L, choose one and label it the active preimage, and the other inactive. Then define $\Delta(H) \in H_1(X;\mathbb{Z}/2\mathbb{Z})$ by:

$$\Delta(H) = \sum_{\substack{c_1 \subset L \\ \text{Type I active}}} \mathrm{lk}(c_1, L - c_1)[g_{c_1}] \quad + \sum_{\substack{g \in \pi_1(X \times I), \\ g^2 = 1,\ g \neq 1}} \left(\sum_{\substack{c_2, c_3 \subset L \text{Type II}, \\ [c_2] = [c_3] = g}} \mathrm{tw}(c_2, c_3)[g_{c_2}] \right)$$

where $[g_{c_i}]$ denotes the homology class of g_{c_i}. The notation $\mathrm{tw}(c_2, c_3)$ denotes the relative twist number of c_2 and c_3; see [15, page 690] for the definition.

Definition 6 (Stong 1994) Given S_0, S_1 homotopic spheres inside X^4 with an immersed dual sphere G such that $G \cdot S_i \equiv 1 \mod 2$ and $\mathrm{fq}(S_0, S_1) = 0$, define the *Kervaire-Milnor invariant* $\mathrm{km}(S_0, S_1)$ by:

$$\mathrm{km}(S_0, S_1) = \Delta(H) \in H_1(X;\mathbb{Z}/2\mathbb{Z})/(\Delta(\mathrm{Self}(S_0)))$$

where $\mathrm{Self}(S_0)$ is the set of immersions of $S^2\times I$ cobounded by two "stacked" copies of S_0 in $X \times I$, and $(\Delta(\mathrm{Self}(S_0)))$ denotes the subgroup of $H_1(X;\mathbb{Z}/2\mathbb{Z})$ generated by $\Delta(H')$ for all H' in $\mathrm{Self}(S_0)$.

Note that in the above definition, the Kervaire-Milnor invariant is defined in this quotient of $H_1(X;\mathbb{Z}/2\mathbb{Z})$ so as to eliminate any contribution from immersed concordances from S_0 and S_1 to themselves. (In fact, it suffices to mod out by only by the Δ invariant of such immersed concordances from S_0, since S_0 and S_1 are homotopic.)

As in the Freedman-Quinn invariant, if $H_3(X;\mathbb{Z}) = 0$, then all immersed self-concordances of S_0 are homologous to $S_0\times I$, and so $\Delta(H') = 0$ for all H' in $\mathrm{Self}(S_0)$. In this case, we need not mod out by $(\Delta(\mathrm{Self}(S_0)))$, and $\mathrm{km}(S_0, S_1) = \Delta(H)$ for any chosen immersion H of $S^2 \times I$.

Δ was first defined by Stong in [15, page 691]. It is constructed so as to be invariant under regular homotopy of an immersed 3-manifold (in Stong's case, B^3) in 5-space, which can change the self-intersection link itself. He used it to prove a version of the following theorem that was then adapted to the concordance setting by Klug and Miller in [8].

Theorem 10 (Klug-Miller) *Let S_0 and S_1 be homotopic spheres that are s-characteristic in X^4 together with an immersed dual sphere G such that $G \cdot S_i \equiv 1$* mod 2. *Then S_0 and S_1 are concordant in X if and only if $km(S_0, S_1) = 0$.*

One should note that $\mathrm{km}(S_0, S_1)$ is only defined for spheres for which $\mathrm{fq}(S_0, S_1)$ vanishes, and only obstructs concordance in the case that S_0 and S_1 are not s-characteristic. If the Freedman-Quinn invariant is understood as measuring the failure of any immersion of $S^2\times I$ to be a concordance because of its image in the fundamental group, the Kervaire-Milnor invariant measures the failure of any such immersion to be a concordance because of the linking of the self-intersection set itself (although the link of self-intersection circles is far from being an invariant of an immersed $S^2 \times I$, even within a given 3-homology class). An example where Kervaire-Milnor invariant does in fact obstruct concordance for a pair of spheres for which the Freedman-Quinn invariant vanishes was shown by Klug and Miller in [8]. Their theorem, below, is proved using a Hopf link configuration of preimages of self-intersection circles.

Theorem 11 *Let $\alpha \in H_1(X;\mathbb{Z}/2\mathbb{Z})$ for some 4-manifold X. Then there exists an embedded sphere S in $X\#\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ such that:*

$$km(\mathbb{CP}^1\#\overline{\mathbb{CP}}^1, S) = \alpha' \in H_1(X\#\mathbb{CP}^2\#\overline{\mathbb{CP}}^2;\mathbb{Z}/2\mathbb{Z})$$

Where α' is the image of α under the inclusion map in the Mayer-Vietoris sequence for the connected sum of X with $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$.

The Kervaire-Milnor invariant remains not very well understood, but quite powerful. For instance, it is open at time of writing whether there exists a 2-sphere S in $S^1 \times B^3$ such that $\mathrm{km}(U, S) \neq 0$ for U the unknotted sphere. The existence of such a sphere would imply the existence of a 2-component link of spheres in S^4 that do not bound disjoint balls in B^5.

3 Exercises

Exercise 1 Why does Kervaire's proof that 2-knots are slice not apply to classical knots? What about maps $S^3 \hookrightarrow S^5$?

Exercise 2 Does every $S^2 \sqcup S^2 \hookrightarrow S^4$ bound a $S^2 \times I \hookrightarrow B^5$?

Exercise 3 Use the Lightbulb Theorem to prove the following: Let D_1, D_2 be disks in S^4 with $\partial D_1 = \partial D_2$. Then D_1 is isotopic to D_2 rel ∂.

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