Guidelines for M175W: Mathematical writing

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Abstract

Guidelines for writing a paper appropriate for M175W are discussed, including information on the permissible styles of the paper, the organization of the paper into sections, the use of language, and the communication of mathematical ideas. A summary of the grading criteria is also given.

1 Introduction

The purpose of M175W is to help students fulfill the writing requirements for the College of Natural Sciences. In addition to enhancing their writing and communication skills, students get an opportunity to do independent study in an area of interest. The student, under the supervision of the instructor, will select a topic and be required to write two papers. Each paper should contain at least 2000 words, which corresponds to about 8-10 pages of ordinary, double-spaced text. The first paper will be due one week before the middle of the semester, and the second one week before the end. Each paper will be graded, and will usually require a rewrite. The course grade will be based on the final versions of the papers, with each paper weighted equally.

The papers can be written in either of two styles: expository or technical. An expository paper should be aimed at an informed, but general audience and should be like an article in a magazine. Its main purpose is to convey general information in an engaging and talkative tone without fussy details. A technical paper should be aimed at a specific audience and should be like an article in a journal or a short chapter in a book. Its main purpose is to communicate technical results and details in a clear, efficient and precise way. For reference, this paper is written in the expository style. A short example of the technical style is given in one section.

Papers will be graded on their structure, language, technical accuracy and mechanics (grammar, spelling and punctuation). In many cases, the paper will require further work in one or more of these areas, and the student will be required to revise it. Revisions will usually be due within a week. The intention of the revisions is to not only improve the paper, but also illustrate the idea that writing is an iterative process. Well-written, published works that appear in magazines, journals and books usually are the result of many iterations. It has been said that good works are never finished, but only abandoned. That is, there will always be room for improvement.

This paper presents a survey of various ideas and opinions on mathematical writing. It contains a summary of some of the basic principles of writing, and provides a model for the basic structure of a paper. It closely follows the paper by Kleiman and Tesler [4], which is used as a writing guide at the Massachusetts Institute of Technology. The objective here is to collect various information that would likely be of use to novice authors, help clarify misconceptions and provide a list of best practices. More detailed discussions of the material outlined here can be found in that work, as well as the works by Alley [1], Flanders [2], Gillman [3], Knuth et. al. [5] and Strunk and White [7].

The presentation is organized as follows. In Section 2, we summarize the basic structure of a paper. We discuss how a paper is broken up into sections and discuss their purpose and content. In Section 3, we discuss the role of language in a paper. We illustrate how language affects the communication of ideas and summarize a list of best practices. In Section 4, we discuss a number of issues that are unique to mathematical writing, such as the treatment of equations and the presentation of formal statements and proofs. In Section 5, we give a sample of technical mathematical writing using the two fundamental theorems of calculus as an example. We state and prove the theorems, and explain their significance. In the Appendix, we discuss special terminology arising in advanced mathematical writing.

2 Organization

A typical paper has a title, abstract and introduction, followed by a number of sections which form the main body, then possibly a conclusion and an appendix, and finally a list of references. Most papers do not have a formal table of contents or index. To help communicate ideas, many papers have figures and tables. Here we describe the purpose and content of these basic elements.

The **title** and **abstract** are arguably the most important elements of a paper. The title should clearly and concisely represent the focus of the work and its most distinctive features. A long, wordy title is very often a turn-off. It has been said that each unnecessary word in a title will result in one less reader. The abstract is essentially a table of contents in paragraph form. It should briefly describe the main contents and results of the paper with little if any detail or background information. Like a trailer for a movie, it should highlight the most important points. A good title and abstract will leave interested readers wanting more, and will leave uninterested readers with time to move on.

The **introduction** is the place where you set the stage for your work. It should clearly state the problem or topic of study, and provide some background and motivation on why the topic is important, and also describe what is known and not known about it, citing relevant literature where appropriate. Most importantly, the introduction should describe your objectives and summarize what the reader will learn from your paper. For example, if your objective is to prove a theorem, then you should describe why your theorem is important and how it might help clarify or provide a deeper understanding of your topic. Alternatively, if your objective is to survey information, then you should describe why this information is important and how it might help clarify frequently asked questions or misconceptions about your topic. A good introduction is one that puts the paper in perspective as a whole. As a matter of taste, introductions usually end with an outline or roadmap of the paper which describes its organization.

The **body** of the paper is where you present your work or findings in detail. It should be broken up into numbered sections with clear, concise titles. Their purpose is to help the reader digest and understand the material. Dividing a paper into sections requires some thought. The author must decide what to put where, what to leave out, and what to emphasize. There is no simple formula for deciding; it depends on the subject and audience. Sections are not necessarily written in chronological order. Very often, a draft of the body of a paper is written first. Then a draft title, abstract and introduction are written to support the body. The process of writing a good introduction may in turn suggest changes to the body, and the whole process repeats. The proverb that good papers are never finished, but only abandoned now becomes clear – the process of revising can continue indefinitely. Notice that the main points of your paper should appear in more than one place. They should be listed in the abstract, described in the introduction and detailed in the body.

A **conclusion** is an optional section which contains additional perspective on the paper and maybe even suggestions for further study. An **appendix** (possibly more than one) is an optional section which gives special detailed information or additional background material for secondary audiences. Conclusion sections are somewhat rare in mathematical writing and are included as a matter of taste. When included, a conclusion should not simply be a mirror image of the introduction; it should add some additional insight that can only be appreciated once the complete story is told. An appendix is usually reserved for supplementary information which is too detailed, tutorial in nature or of only limited interest. Notice that the material in an appendix must be important enough to include in the paper, yet distracting enough to omit from the main body.

The list of **references** is where you put the bibliographical information about each source cited in your paper. There are several different styles for citing sources in the text and for listing bibliographical information in the reference section. The style used in this paper is common in mathematical writing. A citation is indicated by a number in square brackets, where the number indicates the entry in the reference list. Additional information can be included, such as specific page numbers, sections, chapters, etc. For example, citations to the fifth entry in your reference list would be indicated by [5] or [5, p.9] or [5, Chap. 1]. The citation itself is treated somewhat like a parenthetical remark within a sentence. If the citation comes at the end of a clause or sentence, then put the comma or period after the citation, not before it. As illustrated in the reference section of this work, different styles are normally used to list the bibliographical information of different types of sources such as books, articles and online resources.

Figures and **tables** may be included in the main body of a paper to help communicate the main points or results of your work. Each figure and table should have a label, as in Figure 1 and Table 1. Additionally, each figure should have a descriptive caption, and each table a descriptive title. Moreover, each figure and table should be explained, or at the very least mentioned, in the text. Ideally, you should explain what points they illustrate, and what points, if any, they do not. If they are not in complete agreement with other results, then say so and suggest reasons for the discrepancies. Indeed, trying to understand the discrepancies between different results can lead to new and interesting research. Clearly label the parts of an illustration, label the axes of graphs and make sure all symbols used in a figure or table are defined. Place figures and tables closely after their first reference in the text.

3 Language

Ideas are communicated through language consisting of words and sentences. While infinitely many words and sentences can be used to communicate a single idea, some are better than others. Good writing is characterized by language that is precise, clear, concise, familiar, forthright and fluid [1]. These characteristics are not independent; there is much overlap between them. Here we discuss these characteristics within the context of mathematical writing.

Being **precise** means being picky about words. Mathematical ideas are perfect; the language we use to convey them is not. When trying to communicate an idea, take your time and choose your words carefully. Contrary to popular belief, good writing does not require the use of synonyms. Do not be afraid to use the same word more than once. In mathematical writing, the use of a synonym might lead a careful reader to think that the author meant something similar, but perhaps slightly different. A careful reader will be able to tell the difference between an author who is being precise and an author with a small vocabulary. Being precise also means giving specific and relevant details. Beware that a paper which is too detailed can be boring and hard to follow.

Being **clear** means using no unnecessary words or grammatical structures. Being **concise** means keeping sentences simple and to the point. It may help to keep most of them short, but you need some longer ones to keep your writing from sounding choppy and to provide variety and emphasis. Notice that punctuation can be used to help improve clarity and eliminate ambiguities in language. Eliminate needless repetition and small details that the readers can supply for themselves. If something goes without saying, then there is no reason to say it. Clear and concise writing is simple, efficient and beautiful. Each word in each sentence has a purpose; there is no fluff.

Being **familiar** means using language familiar to your readers. Define unfamiliar words, and familiar words used in unfamiliar ways. If the definition is short, then include it in the same sentence, and set it off by commas or parentheses if appropriate. If the definition is complex or technical, then expand it in a sentence or two. Avoid jargon, abbreviations and slang terms. Although such terms may be familiar to you, they might not be familiar to interested readers who are new to your field. Being familiar also means using analogies and everyday language to help explain complicated technical concepts, and describing things in the simplest possible way.

Being **forthright** means being straightforward, honest and friendly. It means using the active rather than passive voice, and writing in the first person present tense. Use the pronoun "we" to refer to the author and reader together, or the author with the reader looking on. The use of "I" is awkward and should be avoided. For instance, the phrase "Here we study a new class of diffusion equations" is generally more desirable than "A new class of diffusion equations is studied" or "Here I study a new class of diffusion equations." The active voice is typically used in all sections of a paper, except possibly the abstract. Indeed, many authors choose to write that section in the passive voice, as it is in this paper.

Being **fluid** means being smooth in transitions from one thought to another. It means avoiding jumps or gaps. To help smoothen transitions, begin each new sentence or thought where the previous one left off. Use connecting words and phrases, and use parallel wording when discussing parallel concepts. Be consistent with voice, person and tense, and pay attention to the length and structure of your sentences. Strive for variety; beware of monotony.

4 Mathematics

Mathematical writing is different from other types of writing in that it often involves equations, either in text or displayed, and formal statements and proofs. Here we discuss these elements and provide some guidelines for writing them effectively.

Equations are mathematical statements which express relations between variables and other mathematical objects. When writing an equation, make sure each variable and symbol is defined either before or immediately after it. It is good practice to explain an equation in simple everyday language so that the reader can get an intuitive feel for it; use analogies if possible. If an equation is valid only under certain assumptions, then it is good practice to say so. If the assumptions are complicated and not of particular importance or relevance to your work, then simply say that and move on. On the other hand, if they are important, then give them their due attention.

A **displayed equation** is one that is centered on a line by itself and usually assigned a numeric label for future reference. An equation which is of particular importance or too long to fit in the text (more than about one-fourth of a line) should be written as a displayed equation. Displayed equations should be punctuated with commas or a period just as an equation in the text would be. A long series of displayed equations should be avoided, especially when only small changes occur from one equation to the next. Instead, it is better to write only the most important equations and describe in words how to get from one equation to the next. Interested readers will be happy to fill in the details for themselves.

A **formal statement** is a statement such as a definition, theorem or corollary. Formal statements should be be given a label such as Definition 1, Theorem 1 or Corollary 1 and written as a separate paragraph that if offset from the regular text. Formal statements should be written as concisely as possible. They should not contain discussion or commentary; that material should appear in the regular text. If references are appropriate, then place them immediately after the label like this: Theorem 1 [5], [7, p.91]. Whether or not a statement is written in formal style is a matter of taste. The formal style is usually reserved for only the most important definitions and assertions. Standard, well-known, easy or straightforward definitions and assertions are usually just described in the text without any formal structure.

A **proof** is a explanation or argument used to establish the validity of an assertion. Any assertion which is not standard or well-known should be accompanied by a proof. If a proof is not of particular importance or relevance to your work, then just provide a reference where one can be found, or consider putting it in an appendix. If a proof can be omitted because it is easy, straightforward or obvious, then simply say so. If the proof is of primary importance to the paper, then it should appear after the assertion in a new paragraph beginning with the one-word sentence "Proof." Conceptual proofs are usually better than computational ones; ideas are easier to communicate and understand. Omit the details of purely routine computations or arguments; highlight any twists or turns. It is good practice to indicate the end of a proof with a symbol such as □, or with the letters QED, which stand for the Latin phrase *quod erat demonstrandum* (which was to be demonstrated).

Good mathematical writing is characterized by various additional practices. For example, symbols such as =, ≤, ∃ and ∀ should be used only in equations, not in the regular text outside of equations. Two or more equations in text should be separated by words like this: $x^2 + y^2 = 1$ and $xy = 1$. If a condition is introduced with "if," then the conclusion should be introduced with "then." Sentences should be started with words, not symbols or equations. Notation should be consistent throughout an entire paper, and should be kept as simple and familiar as possible. Each formal statement such as a theorem should be preceded by a complete sentence, not merely a sentence fragment or introductory clause.

5 Example

As an example of technical mathematical writing, we discuss the two fundamental theorems of calculus as presented in Stewart [6, Chap. 5]. The First Fundamental Theorem says that the process of differentiation is inverse to that of integration. This statement is remarkable because, at first glance, the two processes appear to be unrelated. Indeed, differentiation gives the slope of a curve, whereas integration gives the area under a curve. The precise statement of the theorem is as follows.

Theorem 1 (First Fundamental Theorem of Calculus) *If a function $f(x)$ is continuous on a closed interval $[a, b]$, then the function $g(x)$ defined by*

$$g(x) = \int_a^x f(t)\,dt, \quad a \le x \le b$$

is continuous on $[a, b]$, differentiable on (a, b) and $g'(x) = f(x)$.

Proof. Let x be an arbitrary point in the open interval (a, b) and let $h \neq 0$ be sufficiently small so that $x + h$ is also in (a, b). Then, by definition of $g(x)$ and properties of the integral with respect to its limits, we have

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right] = \frac{1}{h}\int_x^{x+h} f(t)\,dt. \quad (1)$$

Without loss of generality, we assume that $h > 0$. (The case $h < 0$ is similar.) Thus $x + h > x$ and the last integral in (1) is the net area under $f(t)$ in the closed interval $[x, x+h]$. Since $f(t)$ is continuous on this interval, the Extreme Value Theorem [6, Chap. 4] states that $f(t)$ attains absolute minimum and maximum values m, M at some numbers u, v in $[x, x + h]$ as illustrated in Figure 1. Specifically, we have $m = f(u)$ and $M = f(v)$, and by comparison properties of the integral, $mh \leq \int_x^{x+h} f(t)\,dt \leq Mh$. Combining this with (1) we obtain

$$f(u) \leq \frac{g(x+h) - g(x)}{h} \leq f(v). \tag{2}$$

Since $f(t)$ is continuous, and u, v are in the interval $[x, x + h]$, we have $f(u), f(v) \to f(x)$ in the limit $h \to 0$. Moreover, from (2) and the Squeeze Theorem [6, Chap. 2], we deduce

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f(x). \tag{3}$$

Thus $g(x)$ is differentiable and hence continuous at each point x in the open interval (a, b). If $x = a$ or $x = b$, then arguments similar to those above lead to a one-sided limit of exactly the same form as (3). Thus $g(x)$ has one-sided derivatives and hence is also continuous at $x = a$ and $x = b$. ■

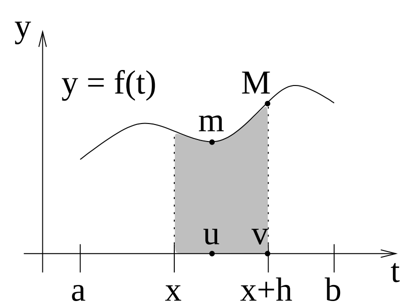

Figure 1: Setup for the proof of the First Fundamental Theorem.

The First Fundamental Theorem states that, given a continuous function $f(x)$, there exists a function $g(x)$ whose derivative is equal to $f(x)$, that is, $g'(x) = f(x)$. Such a function $g(x)$ is called an antiderivative of $f(x)$. It should be clear that antiderivatives are not unique, for if $g(x)$ is an antiderivative, then so is $g(x) + C$ for any constant C. On the other hand, any two antiderivatives can differ by no more than a constant, for if $F(x)$ and $g(x)$ are antiderivatives, then $F'(x) - g'(x) = 0$, which implies that $F(x) - g(x) = C$ for some constant C. When the First Fundamental Theorem is combined with the fact that an antiderivative is unique up to an additive constant, we obtain the following result.

Theorem 2 (Second Fundamental Theorem of Calculus) *If a function $f(x)$ is continuous on a closed interval $[a, b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F(x)$ is any antiderivative of $f(x)$, that is, $F'(x) = f(x)$.

Proof. Let $F(x)$ be given and let $g(x) = \int_a^x f(t)\,dt$. By the First Fundamental Theorem, $g(x)$ is also an antiderivative of $f(x)$, and it follows that $F(x) - g(x) = C$ for some constant C. By substituting for $g(x)$ in this expression, we get

$$F(x) = \int_a^x f(t)\,dt + C. \tag{4}$$

Taking $x = a$ in (4) we find that $C = F(a)$ because $\int_a^a f(t)\,dt = 0$. Taking $x = b$ in (4) and substituting the result for C, we obtain

$$F(b) = \int_a^b f(t)\,dt + F(a). \tag{5}$$

Rearranging this equation and changing the integration variable from t to x yields the desired result. ∎

The Second Fundamental Theorem says that we can evaluate a definite integral of a function merely by subtracting two values of an antiderivative. This is a remarkable statement in view of the definition of the integral in terms of Riemann sums. In practice, antiderivatives are often found by applying differentiation formulas in reverse. A brief list of antiderivatives is given in Table 1, where $\alpha \neq 0$, $\beta \neq 0$ and C are constants. Here we follow standard notation and use the symbol $\int f(x)\,dx$ to denote the most general antiderivative of $f(x)$. We remark that the most general antiderivative is usually referred to as the indefinite integral of $f(x)$.

Table 1: Common antiderivatives

$\int \alpha x^p \, dx = \frac{\alpha}{p+1} x^{p+1} + C \quad (p \neq -1)$
$\int \alpha x^{-1} \, dx = \alpha \ln \lvert x \rvert + C$
$\int \alpha \sin \beta x \, dx = -\frac{\alpha}{\beta} \cos \beta x + C$
$\int \alpha \cos \beta x \, dx = \frac{\alpha}{\beta} \sin \beta x + C$
$\int \alpha e^{\beta x} \, dx = \frac{\alpha}{\beta} e^{\beta x} + C$

Appendix

Key results in advanced mathematical writing are often stated formally as theorems, propositions, corollaries and lemmas. In addition to making your results citable, these formal structures also indicate the nature of the result. Moreover, key terminology and observations are sometimes stated formally as definitions and remarks. Here we discuss some guidelines on the use of these formal structures.

A **theorem** is a major or primary result. It provides a definitive or remarkable answer to a central problem in your work. A **proposition** is an intermediate or secondary result. It is more modest than a theorem, but nevertheless important. Indeed, a typical paper may contain several propositions, all of which lead up to a main theorem. A **corollary** is a minor result which is a straightforward consequence of a theorem, proposition or another corollary. It is a noteworthy observation. A **lemma** is a technical result that is to be used in the proof of some other result, but is not necessarily of independent interest or importance like a proposition. Normally, a lemma is stated and proved before it is used. A **definition** is a formal introduction of some terminology or notation. Only the most important and crucial definitions should be written as formal statements. Standard, well-known, easy or straightforward definitions are usually just described in the regular text. A **remark** is a formal comment or observation which is brief and set apart from the regular text. A group of related remarks is usually presented as an enumerated list.

Various conventions are used in the typesetting of formal statements. The statement is usually offset from the regular text by some extra white space before and after it, and the first line of the statement has no indentation. The heading of the statement is usually typeset in bold, with the exception of a proof, whose heading is usually typeset in italics. The body of the statement is usually typeset as ordinary text, except for theorems, propositions, corollaries and lemmas, whose bodies are usually typeset in italics as illustrated in Section 5.

References

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