Homework 1

1) Let $M \in \mathbb{R}^{3\times 3}$ and $u, v \in \mathbb{R}^3$ be arbitrary.
   (a) Show that $Mu \cdot v = u \cdot M^T v$.
   (b) Show that, if M^{-1} exists, then $(Mu) \times (Mv) = (\det M) M^{-T}(u \times v)$. Recall that $M^{-T} = (M^{-1})^T = (M^T)^{-1}$.

2) For any $Q, R \in \mathrm{SO}_3$ and $u, v \in \mathbb{R}^3$ show:
   (a) $QR \in \mathrm{SO}_3$.
   (b) $(Qu) \cdot (Qv) = u \cdot v$.
   (c) $|Qu| = |u|$.
   (d) $(Qu) \times (Qv) = Q(u \times v)$.

3) (Euler's Theorem). Let $Q \in \mathrm{SO}_3$, $Q \neq I$.
   (a) Use properties of determinants and the fact that $Q \in \mathrm{SO}_3$ to show that $\det(Q - I) = \det(I - Q)$. (Hint: $\det M = \det M^T$).
   (b) Use the result from part (a) to show that the equation $(Q - I)n = 0$ or $Qn = n$ has a solution $n \neq 0$.
   (c) Show that the equation $Qn = n$ has exactly one independent solution $n \neq 0$. In particular, show that if there were a second independent solution m (m independent from n means $m \times n \neq 0$), then $m \times n$ would be a third independent solution, which would then imply $Q = I$ (a contradiction).
   (d) Prove that Q is a simple rotation with axis given by n.

4) Let $\{\boldsymbol{e}_j\}$ be a fixed frame and let $\{\boldsymbol{d}_j\}$ be a second frame defined by $\boldsymbol{d}_1 = \boldsymbol{e}_3$, $\boldsymbol{d}_2 = (\boldsymbol{e}_1 + \boldsymbol{e}_2)/\sqrt{2}$, $\boldsymbol{d}_3 = (\boldsymbol{e}_2 - \boldsymbol{e}_1)/\sqrt{2}$. Moreover, let $D \in \mathbb{R}^{3\times 3}$ be the component matrix for $\{\boldsymbol{d}_j\}$ in $\{\boldsymbol{e}_j\}$. Thus $d_j = De_j$ where $d_j, e_j \in \mathbb{R}^3$ are components in $\{\boldsymbol{e}_j\}$.
   (a) Find the component matrix D.
   (b) Find the axis n and subtended angle $\phi \in [0, \pi]$ associated with D.
   (c) Find the Euler angle coordinates $\Theta^e \in \mathbb{R}^3$ such that $D = \varphi_{\mathrm{euler}}(\Theta^e)$.
   (d) Find the Cayley coordinates $\Theta^c \in \mathbb{R}^3$ such that $D = \varphi_{\mathrm{cayley}}(\Theta^c)$.

5) (Visualizing rotations). Consider the functions

$$D^e(t) = \varphi_{\mathrm{euler}}(\theta^e(t)), \qquad D^c(t) = \varphi_{\mathrm{cayley}}(\theta^c(t))$$

   where $\theta^e(t) = t\Theta^e$, $\theta^c(t) = t\Theta^c$ and $t \in [0, 1]$. Here Θ^e and Θ^c are the Euler and Cayley coordinates of the matrix D from Problem 4.
   (a) Let $d_j^e = D^e e_j \in \mathbb{R}^3$. As t varies in $[0, 1]$ how (qualitatively) should the component vectors d_j^e vary? What about the the component vectors $d_j^c = D^c e_j$?
   (b) Use Matlab to compute $d_j^e(t)$ for $t = 0, 1/20, 2/20, \ldots, 1$. For each value of t plot d_1^e, d_2^e and d_3^e as points in a single 3D plot. Do the final positions of d_j^e agree with the definition of the frame $\{\boldsymbol{d}_j\}$ in Problem 4? (See the example Matlab file *myprogram1.m* at the course webpage for help getting started.)
   (c) Repeat part (b) for $d_j^c(t)$. Does the plot agree with your results from Problem 4? Does it agree with the geometric interpretation of the Cayley map? Are the trajectories of $d_j^c(t)$ the same as $d_j^e(t)$?