M/CAM 393C

Homework 2

1) Let $\omega(t) \in \mathbb{R}^3$ be a continuous function and consider the problem of finding $Q(t) \in \mathbb{R}^{3\times 3}$ such that

$$\dot{Q} = Q[\omega\times], \quad t \in [0,T]. \qquad Q(0) = Q_0. \tag{1}$$

The standard theory for ordinary differential equations guarantees that this equation has a unique continuously differentiable solution $Q(t)$ for each $Q_0 \in \mathbb{R}^{3\times 3}$. Here we show that $Q(t)$ is in fact a rotation matrix whenever Q_0 is.

(a) Let $U(t) = Q(t)^T Q(t)$. Show that if Q satisfies (1) then U satisfies $\dot{U} = U[\omega\times] - [\omega\times]U$ for $t \in [0,T]$ with initial condition $U_0 = Q_0^T Q_0$.

(b) Show that $U(t) \equiv I$ is the solution of the equation in (a) when $U_0 = I$. (Solution is unique).

(c) Use (a) and (b) to deduce that $Q(t) \in \mathrm{SO}_3$ for all $t \in [0,T]$ if and only if $Q_0 \in \mathrm{SO}_3$.

2) Given $\omega(t) \in \mathbb{R}^3$ and $Q_0 \in \mathrm{SO}_3$ consider the initial value problem (1). Here we compare two different numerical methods for approximating the solution $Q(t)$. For concreteness we take $Q_0 = I$, $\omega(t) \equiv \frac{1}{\sqrt{3}}(1,1,1)$ and $T = 2\pi$. Recall that, because $\omega(t) \equiv \omega$ (constant), the exact solution is given by $Q(t) = Q_0 \exp(t[\omega\times])$.

(a) Use the forward Euler method

$$\frac{Q_{n+1} - Q_n}{\Delta t} = Q_n[\omega_n\times] \tag{2}$$

to approximate (1) on the interval $[0,T]$. Using a time step $\Delta t = T/N$ with $N = 50$ compute and visualize the frames defined by Q. To assess the quality of the solution make a plot of $\det Q$ versus t. What happens as time progresses? Does the Euler solution remain in SO_3? Repeat your computations using smaller time steps Δt, say with $N = 100$, 200 and 400. Make a plot or table of the final error $||Q_N - Q(T)||$ versus Δt. Does the error seem to satisfy an estimate of the form $||Q_N - Q(T)|| \approx C\Delta t^r$ for some $r > 0$? Here $||\cdot||$ denotes any matrix norm. (See the example Matlab file *myprogram2A.m* at the course webpage for help getting started.)

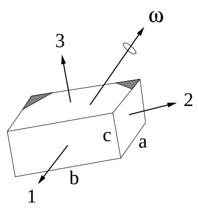
(b) Repeat part (a) using the midpoint method

$$\frac{Q_{n+1} - Q_n}{\Delta t} = Q_{n+\frac{1}{2}}[\omega_{n+\frac{1}{2}}\times], \tag{3}$$

where $Q_{n+\frac{1}{2}} = \frac{1}{2}(Q_{n+1} + Q_n)$ and $\omega_{n+\frac{1}{2}} = \frac{1}{2}(\omega_{n+1} + \omega_n)$. How does the midpoint method compare with forward Euler? Does the midpoint solution remain in SO_3? Does it seem more or less accurate?

(c) In view of the given angular velocity ω, briefly explain why the frame corresponding to Q should make one complete turn about a certain axis in the time interval $[0, 2\pi]$. Do you observe this in your numerical simulations?

3) Consider a uniform rigid block of mass M and dimensions a, b and c. In the absence of a net torque, the rotational motion of the block is described by the system



$$\begin{aligned} \dot{Q} &= Q[\omega\times] \\ \Gamma\dot{\omega} &= (\Gamma\omega) \times \omega, \end{aligned} \tag{4}$$

where $Q \in \mathrm{SO}_3$ is the body frame rotation matrix, $\omega \in \mathbb{R}^3$ are the components of the angular velocity in the body frame, and $\Gamma = \mathrm{diag}(\Gamma_1, \Gamma_2, \Gamma_3) \in \mathbb{R}^{3\times 3}$ is a diagonal matrix of rotational inertia coefficients. Here we study the steady spinning states of this system and their stability.

(a) Given that $\Gamma_1 = \frac{M}{12}(b^2 + c^2)$, $\Gamma_2 = \frac{M}{12}(a^2 + c^2)$, $\Gamma_3 = \frac{M}{12}(a^2 + b^2)$ show that the second equation in (4) can be written in the form

$$\dot{\omega}_1 = \eta_1 \omega_2 \omega_3, \quad \dot{\omega}_2 = \eta_2 \omega_3 \omega_1, \quad \dot{\omega}_3 = \eta_3 \omega_1 \omega_2, \tag{5}$$

where $\eta_1 = \frac{\Gamma_2 - \Gamma_3}{\Gamma_1}$, $\eta_2 = \frac{\Gamma_3 - \Gamma_1}{\Gamma_2}$ and $\eta_3 = \frac{\Gamma_1 - \Gamma_2}{\Gamma_3}$ are dimensionless constants. Moreover, assuming $b > a > c > 0$ show that $\eta_1 < 0$, $\eta_2 > 0$ and $\eta_3 > 0$.

(b) Consider the three non-trivial steady states of (5) derived in class, namely $\omega = (\Omega, 0, 0)$, $(0, \Omega, 0)$ and $(0, 0, \Omega)$ where $\Omega \neq 0$ is the spin rate. Use the Jacobian method to show that $(\Omega, 0, 0)$ is unstable for any $\Omega \neq 0$. Similarly, show that $(0, \Omega, 0)$ and $(0, 0, \Omega)$ are expected to be neutrally stable for any $\Omega \neq 0$.

(c) Let $a = 0.16$-m, $b = 0.24$-m, $c = 0.03$-m, $M = 1$-kg and consider a spin rate of $\Omega = 12$-rad/sec (about two revolutions per second). Approximate (4) on the time interval $t \in [0, 3]$ using the explicit midpoint method with a small time step, say $\Delta t = T/N$ where $N = 1000$. Make and superimpose plots of $\omega_{1,2,3}$ versus t for various initial conditions close to and including $(\Omega, 0, 0)$, say $\omega_0 = (\Omega \pm 0.1, 0 \pm 0.1, 0 \pm 0.1)$. Without loss of generality take $Q_0 = I$. Are the plots consistent with your analysis in part (b)? (See the example Matlab file *myprogram2B.m* at the course webpage for help getting started.)

(d) Repeat part (c) taking initial conditions close to and including the other two steady states. Are your observations consistent with (b) in this case? In particular, do the other two steady states seem neutrally stable?

(e) Find a book with dimensions $b > a > c$ (the more different the better) and verify your results by direct experiment. Try to spin the book in the air about each of its three axes. (You may need a rubberband to keep the book closed). Do you observe the predicted steady states? How do the stability results apply to the experiment? In particular, do you think you can give the book an exact initial angular velocity of the form $(\Omega, 0, 0)$?