Homework 3

1) Consider the equations of motion for a single rigid body subject to a constant gravitational force $\boldsymbol{f}^g$ and a concentrated elastic force $\boldsymbol{f}^c$ at $\boldsymbol{x}$ with energy U. In terms of the usual components we have

$$\dot{q} = v, \qquad \dot{Q} = Q[\omega\times], \qquad M\dot{v} = f^c + f^g, \qquad \Gamma\dot{\omega} = (\Gamma\omega)\times\omega + \tau^c, \tag{1}$$

where $f^c = -\frac{\partial U}{\partial x}$, $\tau^c = Q^T[(x-q)\times f^c]$ and $x = q + Q\xi$ for some fixed body point ξ.

(a) Define the total kinetic energy of the body by $\Psi = \frac{1}{2}v\cdot Mv + \frac{1}{2}\omega\cdot\Gamma\omega \geq 0$. Show that the rate of change of Ψ along any solution of (1) is given by

$$\dot{\Psi} = (f^c + f^g)\cdot v + \tau^c\cdot\omega.$$

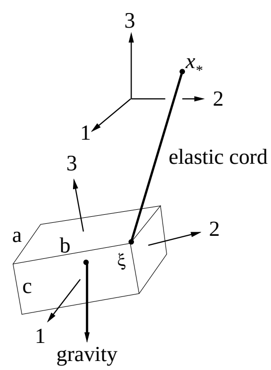
(b) Define the total potential energy of the body by $\Phi = U(x) - f^g\cdot q$, where $x = q + Q\xi$. Show that the rate of change of Φ along any solution of (1) is given by

$$\dot{\Phi} = -(f^c + f^g)\cdot v - \tau^c\cdot\omega.$$

(c) Use (a) and (b) to deduce that the total energy $E = \Psi + \Phi$ must remain constant along any solution of (1).

(d) Consider the case $f^g = 0$ and let $Q = \varphi(\theta)$ be a coordinate map for SO_3. Let (q_*, θ_*) be a local minimum of the elastic energy U so that $(q_*, \theta_*, 0, 0)$ is a steady state for the body. For simplicity assume there is a constant $\alpha > 0$ such that $U(q,\theta) - U(q_*,\theta_*) \geq \alpha(|q-q_*|^2 + |\theta-\theta_*|^2)$ and $\Psi(v,\omega) \geq \alpha(|v|^2 + |\omega|^2)$ for all (q,θ,v,ω). (Thus U and Ψ are "convex" with their global minimum at (q_*,θ_*) and $(0,0)$ respectively). Use the result in (c) to show that the steady state $(q_*,\theta_*,0,0)$ is neutrally stable.

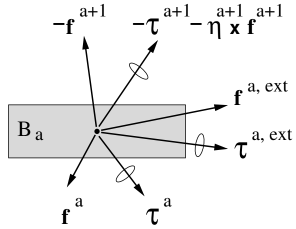
2) Consider a uniform rigid block of mass M and dimensions a, b and c subject to a gravitational force with components $f^g = (0,0,-Mg)$ and a concentrated Hookean spring force with components $f^c = K(|\eta| - \ell)\eta/|\eta|$ where $\eta = x_* - x$ and $x = q + Q\xi$. Here K is the spring stiffness, ℓ is the relaxed length, x_* are the coordinates of the the spring anchor point (fixed frame) and ξ are the coordinates of the spring attachment point (body frame). Recall that $\Gamma = \mathrm{diag}(\Gamma_1, \Gamma_2, \Gamma_3)$ is a diagonal matrix of rotational inertia coefficients where $\Gamma_1 = \frac{M}{12}(b^2+c^2)$, $\Gamma_2 = \frac{M}{12}(a^2+c^2)$, $\Gamma_3 = \frac{M}{12}(a^2+b^2)$.



(a) In kg-m-s units consider the parameters $M = 1$, $a = 0.16$, $b = 0.24$, $c = 0.10$, $g = 10$, $K = 20$, $\ell = 0.25$, $\xi = (\frac{a}{2}, \frac{b}{2}, \frac{c}{2})$ and $x_* = (\frac{a}{2}, \frac{b}{2}, 3c)$. Use the explicit midpoint method to solve (1) on the interval $[0, 10]$ with various different time steps, say $\Delta t = 0.01$, 0.02, 0.025. Use the initial conditions $(q_0, Q_0, v_0, w_0) = (0, I, 0, 0)$. For each different time step make and superimpose plots of the kinetic energy Ψ, potential energy Φ and the total energy E versus t. (See the Matlab file *myprogram3A.m* at the course webpage for help getting started. To use this file you will also need to download the updated version of *plotBox.m*).

(b) Are the numerical results consistent with your analysis in Problem 1abc? In particular, do the slopes of the kinetic and potential energy graphs differ by a minus sign? Does the total energy remain constant? Based on your results, do you think that $\Delta t = 0.025$ is an appropriate time step for simulating this system? What happens for time steps bigger than $\Delta t = 0.025$?

3) Consider a chain of rigid bodies where the net elastic and external loads on a typical body B_a are as follows:



$$
\begin{aligned}
\boldsymbol{f}^a &= \textstyle\sum_i f_i^a \boldsymbol{d}_i^{a-1} \\
\boldsymbol{\tau}^a &= \textstyle\sum_i \tau_i^a \boldsymbol{d}_i^a \\
\boldsymbol{\eta}^a &= \textstyle\sum_i \eta_i^a \boldsymbol{d}_i^{a-1} \\
\boldsymbol{f}^{a,\text{ext}} &= \textstyle\sum_i f_i^{a,\text{ext}} \boldsymbol{e}_i \\
\boldsymbol{\tau}^{a,\text{ext}} &= \textstyle\sum_i \tau_i^{a,\text{ext}} \boldsymbol{d}_i^a .
\end{aligned}
$$

Show that the equations of motion for the usual configuration and velocity variables $(q^a, Q^a, v^a, \omega^a)$ for B_a take the form

$$
\begin{aligned}
\dot{q}^a &= v^a \\
\dot{Q}^a &= Q^a[\omega^a \times] \\
M^a \dot{v}^a &= Q^{a-1} f^a - Q^a f^{a+1} + f^{a,\text{ext}} \\
\Gamma^a \dot{\omega}^a &= (\Gamma^a \omega^a) \times \omega^a + \tau^a - \Lambda^{a+1} \tau^{a+1} - \eta^{a+1} \times f^{a+1} + \tau^{a,\text{ext}},
\end{aligned}
$$

where Λ^{a+1} is the relative rotation matrix and η^{a+1} is the relative displacement between bodies B_a and B_{a+1}, that is, $\boldsymbol{d}_j^{a+1} = \sum_i \Lambda_{ij}^{a+1} \boldsymbol{d}_i^a$ and $\boldsymbol{q}^{a+1} = \boldsymbol{q}^a + \sum_i \eta_i^{a+1} \boldsymbol{d}_i^a$.

4) Consider a rigid body chain with bodies B_a $(a = 1, \ldots, n)$ and quadratic elastic interaction energies

$$
U^a(\eta^a, u^a) = \frac{1}{2} \begin{Bmatrix} \eta^a - \widehat{\eta}^a \\ u^a - \widehat{u}^a \end{Bmatrix}^T \begin{pmatrix} G^a & C^a \\ [C^a]^T & K^a \end{pmatrix} \begin{Bmatrix} \eta^a - \widehat{\eta}^a \\ u^a - \widehat{u}^a \end{Bmatrix},
$$

where $G^a \in \mathbb{R}^{3\times 3}$ is the translational stiffness matrix, $K^a \in \mathbb{R}^{3\times 3}$ is the rotational stiffness matrix, $C^a \in \mathbb{R}^{3\times 3}$ is the coupling matrix and $\widehat{\eta}^a, \widehat{u}^a \in \mathbb{R}^3$ are the relaxed configuration parameters associated with the junction between bodies B_{a-1} and B_a (see figure below). For simplicity we suppose the chain is uniform, so that the energy parameters are independent of the index a.

(a) Consider a chain composed of $n = 20$ uniform blocks, each of mass $M = 0.1$ and dimensions $0.16 \times 0.24 \times 0.10$ along the 1-2-3 body axes. Illustrate the relaxed shape of the chain for each of the following cases:

$$
(\widehat{\eta}, \widehat{u}) = (0, 0, .5,\ 0, 0, 0), \quad (0, 0, .5,\ 0, 0, .17), \quad (.01, 0, .5,\ , 0, .01, .17).
$$

In each case describe whether the chain appears curved/straight and twisted/untwisted.

(b) Consider a chain as in (a) with $n = 10$, relaxed shape $(\widehat{\eta}, \widehat{u}) = (0, 0, .5,\ 0, 0, 0)$, translational stiffness $G = \text{diag}(20, 20, 20)$ and rotational stiffness $K = \text{diag}(3, 3, 3)$. Suppose the last body B_n is subject to a net external force with components $f^{n,\text{ext}} = (0, 0, t)$ for $t \in [0, 7]$. Use the explicit midpoint method to solve the chain equations of motion for each of the following values of the coupling matrix:

$$
C = \text{diag}(0, 0, 0), \quad \text{diag}(0, 0, 1), \quad \text{diag}(0, 0, -1).
$$

In each case use a time step of $\Delta t = 0.01$ and take the relaxed configuration with zero velocities as the initial condition. Illustrate the initial and final shape of the chain in each case. Briefly describe how the coupling matrix C affects the results. (See the Matlab file *myprogram3B.m* at the course webpage for help getting started. This file will also be useful for part (a).)