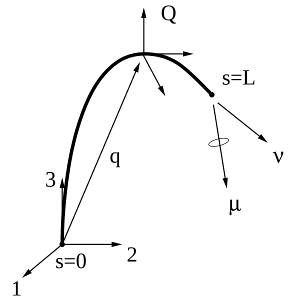
Homework 4

1) Consider the following initial boundary value problem for a filament with configuration components q, Q, velocity components v, ω, mass parameters M, Γ and internal resultant components n, m:



$$\left.\begin{aligned} \dot{q} &= v \\ \dot{Q} &= Q[\omega\times] \\ M\dot{v} &= (Qn)' + f \\ \Gamma\dot{\omega} &= (\Gamma\omega)\times\omega + m' + u\times m + \eta\times n + \tau \end{aligned}\right\}, \quad 0 \le s \le L, \quad t \ge 0,$$
$$\left.\begin{aligned} q(0,t) &= 0, & Q(0,t) &= \mathrm{diag}(1,1,1) \\ n(L,t) &= \nu(t), & m(L,t) &= \mu(t) \end{aligned}\right\}, \quad t \ge 0,$$
$$\left.\begin{aligned} q(s,0) &= q_0(s), & Q(s,0) &= Q_0(s) \\ v(s,0) &= v_0(s), & \omega(s,0) &= \omega_0(s) \end{aligned}\right\}, \quad 0 \le s \le L. \tag{1}$$

Here f, τ are given external load components, ν, μ are given end-load components, and q_0, Q_0 and v_0, ω_0 are given configuration and velocity components. Assume n, m are given by the elastic model

$$n = G[\eta - \widehat{\eta}] + C[u - \widehat{u}], \qquad m = K[u - \widehat{u}] + C^T[\eta - \widehat{\eta}],$$

where G, K, C are the elastic stiffness and coupling matrices, $\widehat{\eta}, \widehat{u}$ are the relaxed shape parameters and η, u are the strain components defined by

$$\eta = Q^T q', \qquad u = \mathrm{vec}(Q^T Q').$$

For concreteness we suppose the material parameters are all constant of the form $G = \mathrm{diag}(G_1, G_2, G_3)$, $K = \mathrm{diag}(K_1, K_2, K_3)$, $C = 0$, $\widehat{\eta} = (0,0,1)$ and $\widehat{u} = 0$. Thus the stress-free configuration is straight, unsheared, untwisted and of length L.

(a) Assuming f, τ and ν, μ are time-independent, show that $(q, Q, v, \omega)(s,t) \equiv (q_0, Q_0, 0, 0)(s)$ is a steady state solution of (1) if and only if (q_0, Q_0) satisfies the boundary value problem

$$\left.\begin{aligned} (Qn)' + f &= 0 \\ m' + u\times m + \eta\times n + \tau &= 0 \end{aligned}\right\}, \quad 0 \le s \le L,$$
$$\begin{aligned} q(0) &= 0, & Q(0) &= \mathrm{diag}(1,1,1) \\ n(L) &= \nu, & m(L) &= \mu. \end{aligned} \tag{2}$$

Any solution (q_0, Q_0) of (2) is called an *equilibrium* or *rest state* of the system.

(b) Consider the case when $\nu = (0,0,c)$, and f, τ, μ are identically zero. Show that $q_0(s) = (0,0,\lambda s)$ and $Q_0(s) \equiv \mathrm{diag}(1,1,1)$ is an equilibrium configuration for an appropriate λ. Sketch the configuration and briefly explain how λ depends on the applied force c and the stiffness constant G_3.

(c) Consider the case when $\mu = (0,0,c)$, and f, τ, ν are identically zero. Show that

$$q_0(s) = (0,0,s), \quad Q_0(s) = \begin{pmatrix} \cos(\Omega s) & -\sin(\Omega s) & 0 \\ \sin(\Omega s) & \cos(\Omega s) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is an equilibrium configuration for an appropriate Ω. Sketch the configuration and briefly explain how Ω depends on the applied twisting torque c and the stiffness constant K_3.

(d) Consider the case when $\mu = (c,0,0)$, and f, τ, ν are identically zero. Show that

$$q_0(s) = r(0, \cos(s/r) - 1, \sin(s/r)), \quad Q_0(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(s/r) & -\sin(s/r) \\ 0 & \sin(s/r) & \cos(s/r) \end{pmatrix}$$

is an equilibrium configuration for an appropriate r. Sketch the configuration and briefly explain how r depends on the applied bending torque c and the stiffness constant K_1.

2) Consider again the initial boundary value problem in (1). Assume the rod undergoes a pure uniaxial motion of the form $q(s,t) = (0,0,z(s,t))$, $Q(s,t) \equiv \mathrm{diag}(1,1,1)$, $v(s,t) = (0,0,\vartheta(s,t))$, $\omega(s,t) \equiv (0,0,0)$, and assume the external loads are of the form $f(s,t) = (0,0,h(s,t))$, $\tau(s,t) \equiv (0,0,0)$, $\nu(t) = (0,0,c(t))$, $\mu(t) \equiv (0,0,0)$. Show that (1) reduces to the following one-dimensional problem for the axial position variable $z(s,t)$:

$$\left.\begin{array}{lll} \dfrac{\partial^2 z}{\partial t^2} = \alpha^2 \dfrac{\partial^2 z}{\partial s^2} + g, & & 0 \le s \le L, \quad t \ge 0 \\[2ex] z(0,t) = 0, & \dfrac{\partial z}{\partial s}(L,t) = \beta(t), & t \ge 0 \\[2ex] z(s,0) = z_0(s), & \dfrac{\partial z}{\partial t}(s,0) = \vartheta_0(s), & 0 \le s \le L. \end{array}\right\}$$

Here $\alpha > 0$, β and g are quantities which you should determine. The partial differential equation for z is the simplest example of a *wave equation*. In general, it admits wave-like solutions which travel at speed α.