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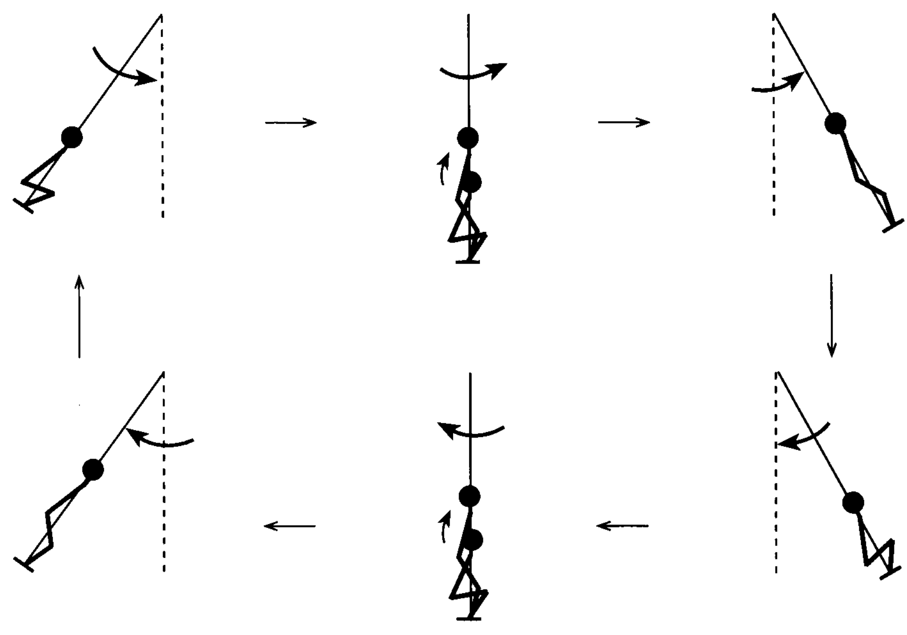

Strategy for Pumping a Swing while Standing

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How to Pump a Swing

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We model a playground swing as a slightly elaborated simple pendulum. In this model, the term *pumping* describes a repeated change in the rider's position and/or orientation relative to the suspending rope or rod. Pumping can gradually increase the amplitude of the swinging. We will describe two ideal models for pumping a swing, based on observations of schoolchildren at play [**13**] and on ideas from 30 years of papers in the *American Journal of Physics* [**1–6, 9–11**].

Our idealization of *pumping from a standing position* is that the rider crouches at the high point of a swing and then suddenly stands up as the swing passes its lowest point. The rider is thus effectively lengthening the pendulum at the high points of the swing and shortening the pendulum at the low points. This can be done both on the forward and on the return cycle. The analysis will show that on each crouch–stand cycle the swing gets a boost in energy from the rider.

Our idealization of *seated pumping* is quite different. It is based on a sudden rotation of the rider's body with respect to the support rope, when the swing momentarily comes to a stop at its highest points. These rotations directly increase the amplitude of the swing's oscillation by lifting the rider slightly above the previously highest level.

After modeling these two idealized pumping strategies using differential equations, we will compare them. Our main conclusion is that seated pumping is the better strategy at low amplitudes, but above a certain amplitude standing is more effective.

This will come as no surprise to experienced swing riders, but the argument provides a nice example of how modeling and qualitative analysis with differential equations can lead to a deeper understanding of a familiar system.

Pumping from a Standing Position

The pumped swing is modeled as a pendulum with variable length L. The rider is modeled as a point mass m, and L is the distance from the rider's center of mass to the fixed swing support point O. Conservation of angular momentum for a point mass undergoing plane motion is

$$\frac{dH}{dt} = N, \tag{1}$$

where H is the angular momentum of the body about O, and N is the net torque about O due to all forces acting on the point mass [7]. In our case H is $mL^2\dot{\phi}$ and the torque about O due to the gravitational force is the product of the transverse component of this force, $-mg\sin\phi$, and the lever arm L. The torque about O due to the tension T in the ropes is 0 (Figure 1). Thus, after dividing out the mass, we have the equation of motion

$$\frac{d}{dt}\left(L^2\dot{\phi}\right) = -gL\sin\phi. \tag{2}$$

As the rider stands or crouches, the effective length L of the pendulum varies with time; however, as the rider does not decide when to change his position by looking at his watch, the length of the pendulum is not well modeled as an explicit function of time $L = L(t)$. Instead, the rider modifies his stance in accordance with the position and velocity of the pendulum. Therefore, we model the pendulum length as an *autonomous* function of the state of the pendulum; that is, $L = L(\phi, \dot{\phi})$.

Following Tea and Falk [**11**], we assume that the rider squats for the first half of each swing of the pendulum, then suddenly stands up as the swing passes through $\phi = 0$ and remains standing for the upward part of the motion. When the pendulum reaches its maximum height and comes to rest instantaneously, the rider again squats and repeats the forcing cycle, this time with the swing moving in the opposite direction (Figure 2).

Suddenly standing up causes a decrease in L, and squatting causes an equal increase in L. We mathematically model the decrease in L when $\phi \approx 0$ as beginning with the rider in a squatting position when $t = t_0$ and ending at a time Δt later with the rider standing upright, where $|\phi| \le \epsilon$ for $t_0 \le t \le t_0 + \Delta t$. Integrating equation

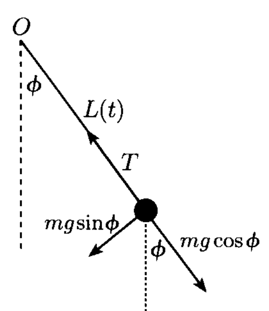

Figure 1. Geometry and free-body diagram of a swing pumped from the standing position.

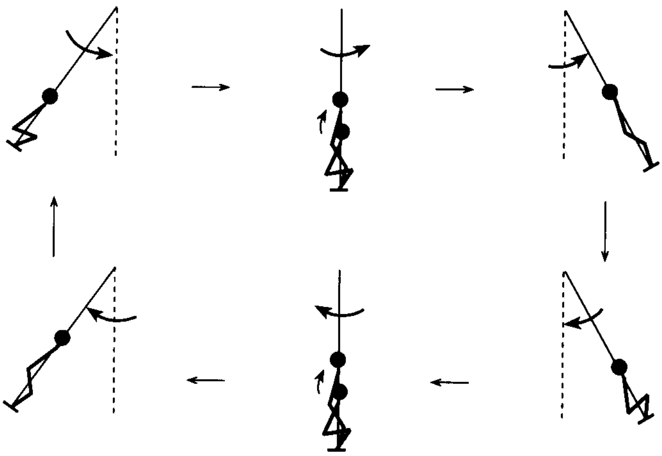

Figure 2. Pumping strategy for a standing rider.

(2) over $[t_0, t_0 + \Delta t]$ gives

$$L^2_{\text{stand}}\,\dot\phi_{\text{stand}} - L^2_{\text{squat}}\,\dot\phi_{\text{squat}} = -\int_{t_0}^{t_0+\Delta t} gL\sin\phi\,dt.$$

Since $|\sin\phi| \le \epsilon$, the integral on the right-hand side is $O(\epsilon)$, so as $\epsilon \to 0$ we have

$$L^2_{\text{stand}}\,\dot\phi^+ - L^2_{\text{squat}}\,\dot\phi^- \to 0.$$

To simplify the notation for the angular velocities just before and just after the decrease in L caused by the rider's suddenly standing up, we write $\dot\phi^-$ for $\dot\phi_{\text{squat}} = \dot\phi(t_0)$ and $\dot\phi^+$ for $\dot\phi_{\text{stand}} = \dot\phi(t_0 + \Delta t)$. Thus the rider's standing up as ϕ passes through 0 produces the following boost in the angular velocity $\dot\phi$:

$$\dot\phi^+ = \left(\frac{L_{\text{squat}}}{L_{\text{stand}}}\right)^2 \dot\phi^-. \tag{3}$$

What is the effect of the increase in L caused by returning to a squat when $\dot\phi \approx 0$ at the high points in the motion? Assume the rider is standing at some time t_1, and a bit later at time $t_1 + \Delta t$ the rider is fully crouched, with $|\dot\phi| \le \epsilon$ for $t_1 \le t \le t_1 + \Delta t$. Then the values of ϕ before and after the rider returns to a crouch differ by no more than 2ϵ. In the limit, therefore, as $\epsilon \to 0$, returning to the squatting position produces no increase or decrease in the angular velocity, and hence in the kinetic energy, of the pendulum. To see whether the swing angle is affected, we integrate equation (2) from t_1 to some time $t \le t_1 + \Delta t$ and obtain

$$L^2(t)\dot\phi(t) - L^2_{\text{stand}}\,\dot\phi(t_1) = -\int_{t_1}^{t} gL\sin\phi\,d\tau.$$

Dividing by $L^2(t)$ and integrating again, this time from t_1 to $t_1 + \Delta t$, gives

$$\phi(t_1 + \Delta t) - \phi(t_1) - \int_{t_1}^{t_1+\Delta t} \left(\frac{L_{\text{stand}}}{L(t)}\right)^2 \dot\phi(t_1)\,dt = \int_{t_1}^{t_1+\Delta t} \frac{-1}{L^2(t)} \int_{t_1}^{t} gL\sin\phi\,d\tau\,dt.$$

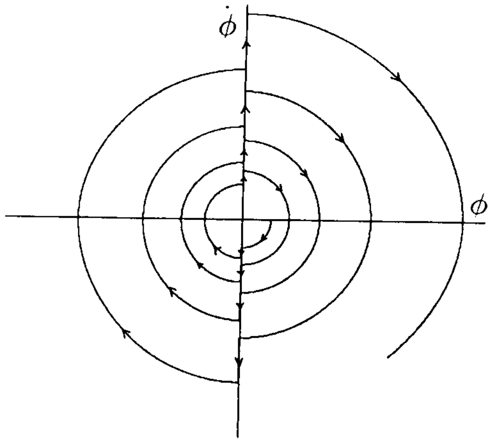

Figure 3. Phase trajectory for standing pumping. $L_{\text{stand}} = 2.3$, $L_{\text{squat}} = 2.7$, $g = 9.8$.

The integral on the left is $O(\epsilon)$ and that on the right is $O\big((\Delta t)^2\big)$; thus, in the limit as Δt and ϵ approach 0, we have $\phi^+ = \phi^-$. So, as physical intuition might suggest, the rider's sudden squatting motion does not affect the angular velocity or the angle of the swing. This conclusion is confirmed in Figure 3, which shows the results of a numerical integration of equation (2) for a rider following the strategy indicated in Figure 2.

In each forcing cycle, the swing gains some energy when the rider rises from a squat as the swing passes through its vertical position. When the reverse move is made, however, from standing to squatting, which occurs when the swing has 0 angular velocity, some potential energy is lost, but it is not as much as the potential and kinetic energy gain from standing up.

Pumping from a Sitting Position

In our idealization of pumping a swing while sitting down, the seated rider changes the orientation of her body with respect to the rope, causing an increase in system energy. This may change the distance from the support point O to the rider's center of mass, but in our current model we will ignore this effect and consider only the effect produced by rotation of the rider's body. We model the seated rider as a barbell of fixed length $2a$ attached at its center to a pendulum arm of constant length L, with the barbell (representing the rider's body) making a variable angle θ with the current direction of the pendulum arm (Figure 4).

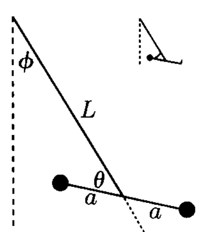

Figure 4. Model of a swing pumped from the seated position.

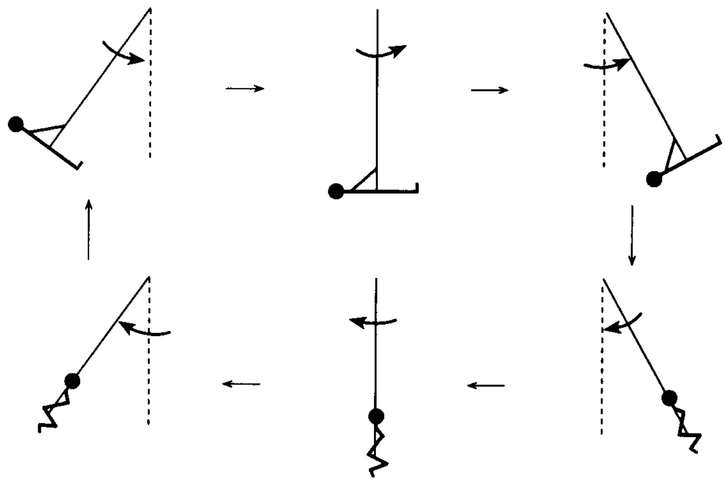

Figure 5. Strategy for pumping while seated.

The governing differential equation in this case is

$$\frac{d}{dt}\left[\left(L^2+a^2\right)\dot{\phi}+a^2\dot{\theta}\right]=-gL\sin\phi. \tag{4}$$

This equation may be derived using (1) by writing the angular momentum as a sum of two terms: the angular momentum of the system concentrated at the center of mass, $mL^2\dot{\phi}$, plus the angular momentum of the system about its center of mass, $ma^2(\dot{\phi}+\dot{\theta})$.

As before, we must look for an autonomous forcing function $\theta(\phi,\dot{\phi})$. Observing children at a local school playground led us to the following strategy for pumping a swing while seated. At the start of the forward motion, when the swing is at its highest point, the rider throws her head back and feet forward. She holds this position until the swing reaches its opposite extreme position and comes to rest instantaneously, so $\theta\approx 90°$ while $\dot{\phi}>0$. Then, before the swing begins its return trip, she pulls her head and torso forward and tucks her feet under the seat, holding this position for the entire return trip, so $\theta\approx 0$ while $\dot{\phi}<0$. Then the forcing cycle is repeated; see Figure 5.

We mathematically model the jump in θ from 0° to 90° as beginning just after time 0 and ending by time Δt, during which time interval $|\dot{\phi}|<\epsilon$. Integrating (4) from 0 to $t\leq\Delta t$ gives

$$\left(L^2+a^2\right)\left[\dot{\phi}(t)-\dot{\phi}(0)\right]+a^2\left[\dot{\theta}(t)-\dot{\theta}(0)\right]=-gL\int_0^t\sin\phi\,d\tau.$$

We then integrate again, from 0 to Δt, and obtain

$$\left(L^2+a^2\right)\left[\phi(\Delta t)-\phi(0)-\dot{\phi}(0)\Delta t\right]+a^2\left[\theta(\Delta t)-\theta(0)-\dot{\theta}(0)\Delta t\right]=\\-gL\int_0^{\Delta t}\int_0^t\sin\phi\,d\tau\,dt.$$

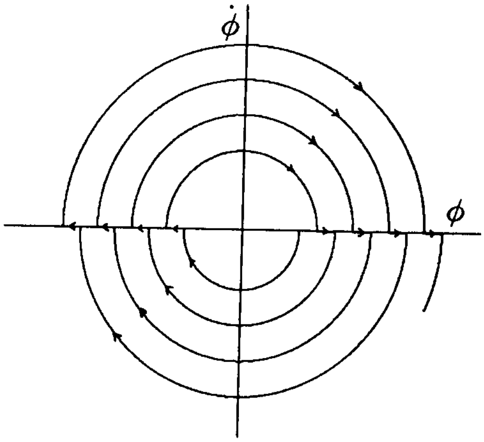

Figure 6. Phase trajectory for seated pumping. $L = 2.5$, $a = 0.5$, $g = 9.8$.

The integral on the right side is $O\left((\Delta t)^2\right)$, and we know that $|\dot\phi(0)| < \epsilon$ and $\dot\theta(0) = 0$, so we conclude that in the limit, as Δt and ϵ approach 0,

$$\left(L^2 + a^2\right)\Delta\phi = -a^2\Delta\theta.$$

Thus, by suddenly increasing her body angle θ by $\pi/2$ at the start of the forward motion, just when the swing is at its highest point, the rider increases the maximum angular displacement of the pendulum from the previous maximum value $\phi(0)$, a negative angle, to $\phi(0) - a^2\pi/\left[2\left(L^2 + a^2\right)\right]$, a negative angle with larger absolute value. Denoting the previous amplitude $|\phi(0)|$ by ϕ^- and the new amplitude by ϕ^+, we have

$$\phi^+ - \phi^- = \frac{a^2\pi}{2\left(L^2 + a^2\right)}. \tag{5}$$

When the rider pulls herself upright at the start of the return motion, so $\Delta\theta = -\pi/2$ and $\phi(0) > 0$, we have $\phi(\Delta t) = \phi(0) + a^2\pi/\left[2\left(L^2 + a^2\right)\right]$, so again the amplitude is increased in accordance with (5). That is, on each half-cycle to or fro the rider's sudden rotation of her body increases the amplitude of the swing by an approximately constant amount (Figure 6).

The Two Pumping Styles Compared

To compare standing pumping with seated pumping, we need to find how much the amplitude increases on each cycle.

During any time intervals when the standing rider's position is unchanged, energy is conserved; thus

$$\frac{1}{2}L^2\dot\phi^2 - gL\cos\phi = \text{constant}. \tag{6}$$

Recall that our rider's strategy is to stand up when ϕ reaches 0 and then squat when $\dot\phi$ reaches 0. We apply (6) to a time interval from just after the rider stands up to just before she squats:

$$\frac{1}{2}L_{\text{stand}}^2 0^2 - gL_{\text{stand}}\cos\phi^+ = \frac{1}{2}L_{\text{stand}}^2\left(\dot\phi^+\right)^2 - gL_{\text{stand}}\cos 0,$$

where ϕ^+ represents the amplitude of the motion after the rider stands up. Dividing by gL_{stand}, we obtain

$$1 - \cos\phi^+ = \frac{L_{\text{stand}}}{2g}\left(\dot{\phi}^+\right)^2. \tag{7}$$

The amplitude that would have been achieved if the rider had not stood up, which we denote by ϕ^-, is found in the same way by applying (5) to the interval starting as the rider crouched, when $\phi = 0$, and ending just before she stood up, when $\dot{\phi} = 0$:

$$1 - \cos\phi^- = \frac{L_{\text{squat}}}{2g}\left(\dot{\phi}^-\right)^2. \tag{8}$$

Dividing (7) by (8) gives us

$$\frac{1-\cos\phi^+}{1-\cos\phi^-} = \frac{L_{\text{stand}}\left(\dot{\phi}^+\right)^2}{L_{\text{squat}}\left(\dot{\phi}^-\right)^2} = \left(\frac{L_{\text{squat}}}{L_{\text{stand}}}\right)^3, \tag{9}$$

where we have used (3). If we approximate $\cos\phi$ by $1 - \phi^2/2$ in (9), we obtain

$$\phi^+ \approx \phi^- \left(\frac{L_{\text{squat}}}{L_{\text{stand}}}\right)^{3/2}. \tag{10}$$

Comparison of (10) and (5) shows that standing pumping *multiplies* the swing's current amplitude by a factor larger than 1, whereas seated pumping *adds* a fixed positive quantity to the current amplitude. Thus for small amplitudes, seated pumping is more effective, but for larger amplitudes standing pumping works better.

A Combined Model

Since people have upper bodies more massive than their lower legs, let's add realism by attaching the barbell representing the rider's body to the pendulum arm at a point away from the barbell's center. If we then apply a strategy of body rotation, we have a model that includes effects of both rotational motions of the rider and changes in the effective length of the pendulum. A similar model was proposed by Case and Swanson [**3**], but they assumed a nonautonomous forcing function $\theta = \theta_0 \cos\omega t$. We define θ and ϕ as in the previous model, and the governing differential equation is

$$\frac{d}{dt}\left[(I_1 + I_2)\,\dot{\phi} + I_2\dot{\theta} - (LN\cos\theta)(\dot{\theta} + 2\dot{\phi})\right] = -2Lg\sin\phi + Ng\sin(\phi+\theta) \tag{11}$$

where $I_1 = 2L^2$, $I_2 = b^2 + c^2$, and $N = c - b$, and where b and c are the distances from the ends of the barbell to its attachment point. A derivation of this equation can be found in [**3**]. Note that (11) reduces to (4) in the case when $a = b = c$. Again we use an autonomous function to describe the angle $\theta = \theta(\phi, \dot{\phi})$. In this combined model, a change of θ when $\dot{\phi} = 0$ causes a direct change in the amplitude, as in our simplified model of seated pumping, as well as an effective change in L because the distance from the support to the center of mass of the rider now depends on θ.

We will not give the details of the analysis here; instead, we will simply describe the two natural pumping strategies and the results of numerical simulation of the model. Our earlier strategy for pumping from a sitting position—setting $\theta = 90°$ when $\dot{\phi} > 0$ and setting $\theta = 0°$ when $\dot{\phi} < 0$—produces a boost in ϕ at either extreme point, as

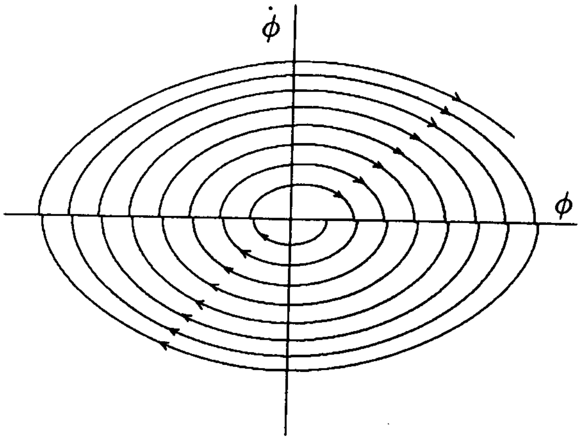

Figure 7. Phase trajectory for seated rider in the 1:1 pumping strategy. $L = 2.5$, $b = 0.4$, $c = 0.6$, $g = 9.8$.

in the earlier symmetric barbell model (Figure 7). Since this strategy produces one boost in the amplitude with each to or fro swing of the pendulum, we may call it the 1:1 pumping strategy.

A combination of the sitting–pumping and standing–pumping strategies is also possible with this model:

1. Set $\theta = 90°$ on the first half of each forward swing, and suddenly change to $\theta = 0°$ as the pendulum passes through $\phi = 0$.
2. On the return swing, set $\theta = 90°$ at the highest point and maintain this for the first half-swing, then again suddenly change to $\theta = 0°$ as the pendulum passes through $\phi = 0$ (Figure 8).

This strategy gives a jump in the angular velocity $\dot{\phi}$ when $\phi = 0$, because the rider's sitting up at these times decreases the effective length of the pendulum just as

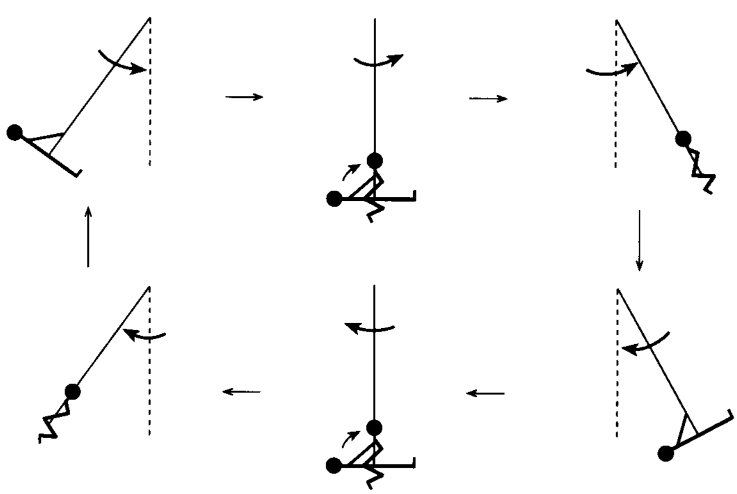

Figure 8. The 2:1 pumping strategy for a seated rider.

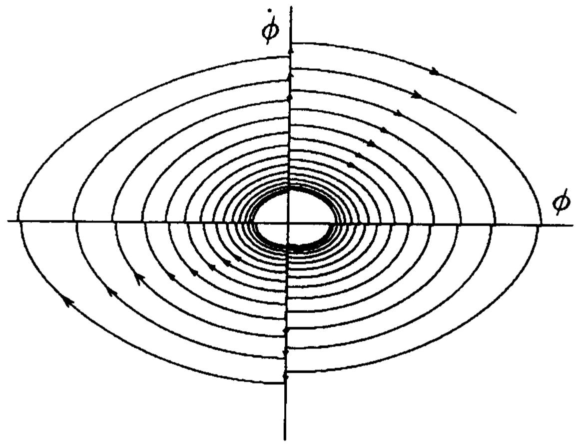

Figure 9. Phase trajectory for seated rider in the 2:1 pumping strategy. $L = 2.5$, $b = 0.4$, $c = 0.6$, $g = 9.8$.

standing up would, only by a lesser amount. Numerical simulation of the off-center barbell model with a rider using this 2:1 strategy shows that the sudden rotation of the rider's body to $\theta = 90°$, when $\dot{\phi} = 0$ at the start of the forward motion, causes a small increase in the angular displacement ϕ, just as it did in the symmetric barbell model (Figure 9). But employing this same rotation as the backward motion begins (when ϕ is at its local maximum) now produces a small *decrease* in ϕ. This is to be expected, because this rotation is the opposite of that used by the seated rider in the symmetric model at the corresponding time. Formerly, the rider pulled herself up into an upright position at this time (Figure 5); but now she lies back, so her rotational motion now decreases ϕ a bit.

Numerical simulation confirms another result that our earlier analysis leads us to expect: The 1 : 1 strategy works best at small amplitudes, while the 2 : 1 strategy becomes more efficient once the amplitude has grown sufficiently large. We find switching from 1:1 to 2:1 pumping while seated in a real swing awkward, but we are optimistic that such a strategy could provide large amplitude motions. Perhaps some energetic readers will try this out in their local playground and let us know!

Suggestions for Further Research

The models of swing pumping discussed here leave many possibilities for further investigations. Here are a few.

Exercise 1. Find the amplitude at which it is optimal for a given rider to switch from 1:1 to 2:1 forcing, in the off-center barbell model.

Exercise 2. We have neglected damping in this work. Physical sources of dissipation could include both air resistance and friction in the swing support. Using numerical integration, explore the effects of damping: Will it limit the maximum amplitude of oscillation for a given pumping strategy?

Exercise 3. Evaluate the effects of non-rigid swing supports (for example, braided nylon ropes hung from a flexible tree branch) by making a model that includes these features and comparing it to the rigid support model [**12**].

Exercise 4. Investigate the energy cost of these strategies. Although you may not be tired after swinging, you did expend some energy to pump the swing. And even though the net work of pumping is equal to the net change in the swing energy, some of the pumping work involves absorbing energy. But people's muscles (unlike generators) cannot store up work that goes into them. Although the details of muscle energetics are not well understood [**8**], a reasonable question to explore might be: Which strategy gives the biggest change in amplitude per unit of positive work?

Conclusion. We have described two kinds of swing pumping strategies. For the idealized standing mechanism, the energy increase comes from the extra work of standing when the centripetal acceleration is high. It, and its modification in the 2:1 strategy for seated pumping, increases the amplitude of the swing in proportion to the present amplitude and thus leads to a *geometric* increase in swing amplitude. For standard 1:1 seated pumping, the energy input comes from work against gravitational potential energy at the swing endpoints. The amplitude increase per cycle is independent of amplitude, so it leads to an *arithmetic* increase in swing amplitude.

The ideal sitting strategy we proposed requires the transmission of a torque by the rope. But ropes are generally idealized as being incapable of carrying a torque. To be purists, we could say we are modeling swings that are suspended by rigid rods, but this would negate our claim to any playground realism. One can think of the rope or rod as merely a mediator of force and torque between the rider and the top hinge. Although we have not made the comparison in detail, we believe that the visible kinking of a playground swing rope and the consequent rotation of the tension force are reasonably approximated by the application of a torque and sideways force to the end of a rigid rod.

Real swinging, both standing and sitting, involves a combination of the two mechanisms described in this paper. Although our model leaves many questions about real swinging unanswered, it does explain the tendency of real swing riders to switch from sitting to standing, to achieve high amplitude.

Acknowledgment. We thank Steve Strogatz and John Hubbard for suggesting this problem, and Joe Burns for helpful comments.

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