Determinants Under Row Operations

Let the matrices A and B be defined as follows:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

1. Calculate $|A|$ and $|B|$.

   Solution:

   By the definition of a 2×2 determinant, $|A| = 1 \cdot 1 - 1 \cdot 2 = \boxed{-1}$. Using row expansion along the first column to calculate $|B|$,

   $$\begin{aligned} |B| = \begin{vmatrix} 0 & 1 & 3 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix} &= 0 \cdot \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} \\ &= -1(-3) + 1(-1) = \boxed{2} \end{aligned}$$

2. Calculate $|R(A)|$ if R is the row operation Row $1 \to 2 \times$ Row 1.

   Solution:

   Clearly,

   $$R(A) = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$$

   Therefore, $|R(A)| = 2 \cdot 1 - 2 \cdot 2 = \boxed{-2}$.

3. Calculate $|R(B)|$ if R is the row operation Row $2 \to (-1) \times$ Row 2.

   Solution:

   Calculating $R(B)$ just like above,

   $$\begin{aligned} |R(B)| = \begin{vmatrix} 0 & 1 & 3 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} &= 0 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} \\ &= -1(3) + 1(1) = \boxed{-2} \end{aligned}$$

4. Make a conjecture about the effect of scalar multiplication on the determinant: that is, if R is the row operation Row $i \to c \times$ Row i, what is the relationship between $|C|$ and $|R(C)|$ for any matrix C?

   Solution:

   As can be conjectured, $|R(C)| = c|C|$. (Sorry for the 'c' overload!)

5. Calculate $|R(A)|$ if R is the row operation Row 2 $\to$ Row 2 $-$ 2 $\times$ Row 1.

   Solution:

   Just like before,

$$|R(A)| = \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = \boxed{-1}$$

6. Calculate $|R(B)|$ if R is the row operation Row 1 $\to$ Row 1 $-$ Row 3.

   Solution:

   Expanding along the second column,

$$|R(B)| = \begin{vmatrix} -1 & 0 & 3 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix} = -0 \cdot \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} + 0 \cdot \begin{vmatrix} -1 & 3 \\ 1 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} -1 & 3 \\ 1 & -1 \end{vmatrix}$$
$$= -1(1-3) = \boxed{2}$$

7. Make a conjecture about the effect of adding a scalar multiple of a row on the determinant: that is, if R is the row operation Row $i \to$ Row $i + c \times$ Row j, what is the relationship between $|C|$ and $|R(C)|$ for any matrix C?

   Solution:

   As is demonstrated above, this kind of row operation doesn't change the determinant. Hence,

$$|R(C)| = |C|$$

8. Calculate $|R(A)|$ if R is the row operation (Swap Row 1 and Row 2).

   Solution:

   Like above,

$$|R(A)| = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = \boxed{1}$$

9. Calculate $|R(B)|$ if R is the row operation (Swap Row 1 and Row 3).

Solution:

Expanding along the first column,

$$|R(B)| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{vmatrix} = 1 \cdot \begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} + 0 \cdot \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$
$$= 1(1) - 1(3) = \boxed{-2}$$

10. Make a conjecture about the effect of swapping two rows of a matrix: that is, if R is the row operation (Swap Row i and Row j), what is the relationship between $|C|$ and $|R(C)|$ for any matrix C?

Solution:

As can be seen from above,

$$|R(C)| = -|C|$$

11. Now, use the answers from 4, 7 and 10 to fill in the following table:

Row Operation	Effect
Row $i \to c \times$ Row i	Determinant is multiplied by c: that is, $\lvert R(C)\rvert = c\lvert C\rvert$
Row $i \to$ Row $i + c \times$ Row j	Determinant doesn't change: that is, $\lvert R(C)\rvert = \lvert C\rvert$
Swap Row i and Row j	Determinant switches sign: that is, $\lvert R(C)\rvert = -\lvert C\rvert$

Determinants Of Upper Triangular Matrices

Find the determinants of the following upper triangular matrices: (hint: use the easiest column to expand along for the 3×3 matrices!)

1. $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

   Solution:

$$\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = 2 \cdot 3 - 1 \cdot 0 = \boxed{6}$$

2. $\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$

   Solution:

$$\begin{vmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot 0 = \boxed{a_{11}a_{22}}$$

3. $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix}$

   Solution:

   Expanding along the first column,

$$\begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} - 0 \cdot \begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} + 0 \cdot \begin{vmatrix} -1 & 2 \\ 2 & 3 \end{vmatrix} = \boxed{-2}$$

4. $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$

   Solution:

   Expanding along the first column,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ 0 & a_{33} \end{vmatrix} - 0 \cdot \begin{vmatrix} a_{12} & a_{13} \\ 0 & a_{33} \end{vmatrix} + 0 \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{33} \end{vmatrix}$$
$$= \boxed{a_{11}a_{22}a_{33}}$$

5. Make a conjecture about the determinant of a $n \times n$ upper triangular matrix with the entries $a_{11}, a_{22}, \ldots, a_{nn}$ on the diagonal.

   Solution:

   As should be apparent from above, this is equal to $\boxed{a_{11}a_{22}\cdots a_{nn}}$ – that is, the product of the entries on the diagonal.

6. Now, prove your conjecture for Question 5 using induction.

   Hint: This proof should be very similar to the calculations above – just expand along the first column!

   Solution:

   First, a little bit of discussion before the actual proof. Here, the nth statement is: "For any $n \times n$ upper triangular matrix A with entries a_{ij}, $|A| = a_{11}a_{22}\cdots a_{nn}$." An inductive argument does two things: it proves the *base case* – that is, the case correspond to the smallest value of n, and then it proves the statement for $n = k+1$ *assuming* the statement for $n = k$. Let's proceed!

   Proof:

   Base case:
   Here, we show that the statement holds for $n = 1$.

   Asssume: A is a 1×1 upper triangular matrix with entries a_{ij}.
   Need to show: $|A| = a_{11}$.

   A 1×1 matrix A with entries a_{ij} is just $[a_{11}]$. By definition, $|A| = a_{11}$, which is precisely what we wanted.

   Inductive step:
   Here, we show that the statement for $n = k$ implies the statement for $n = k+1$.

   Asssume: For any $k \times k$ upper triangular matrix A with entries a_{ij}, $|A| = a_{11}a_{22}\cdots a_{kk}$.
   Need to show: For an $(k+1) \times (k+1)$ upper triangular matrix A with entries a_{ij}, $|A| = a_{11}a_{22}\cdots a_{(k+1)(k+1)}$.

Let A be a $(k+1) \times (k+1)$ upper triangular matrix with entries a_{ij}. For ease of visualization, that means that:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(k+1)} \\ 0 & a_{22} & \cdots & a_{2(k+1)} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{(k+1)(k+1)} \end{bmatrix}$$

Now, to figure out the determinant of A, expand along the first column:

$$\begin{aligned} |A| &= a_{11}|A_{11}| - a_{21}|A_{21}| + \cdots + (-1)^{k+2} a_{(k+1)1}|A_{(k+1)1}| \\ &= a_{11}|A_{11}| + 0 + \cdots + 0 = a_{11}|A_{11}| \end{aligned}$$

Now, it should be clear that A_{11} is a $k \times k$ upper triangular matrix: to be precise,

$$A_{11} = \begin{bmatrix} a_{22} & a_{23} & \cdots & a_{2(k+1)} \\ 0 & a_{33} & \cdots & a_{3(k+1)} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{(k+1)(k+1)} \end{bmatrix}$$

Therefore, our assumption (often called the *inductive hypothesis*) tells us that $|A_{11}|$ can be calculated by simply multiplying the entries on the diagonal. To be precise, $|A_{11}| = a_{22}a_{33} \cdots a_{(k+1)(k+1)}$. Plugging that into the formula above gets that

$$|A| = a_{11}(a_{22} \cdots a_{(k+1)(k+1)}) = a_{11}a_{22} \cdots a_{(k+1)(k+1)}$$

which is precisely what we wanted to show! □

Algorithm For Calculating the Determinant Using Row Operations

Now use the results from the last two sections to suggest an algorithm for calculating the determinant in an efficient way using row operations:

Algorithm:

1. First, use row operations to bring the matrix into upper triangular form, recording what each row operation does to the determinant.
2. Then, "undo" the operations recorded in Step 1 to get the determinant of the original matrix.

Determinants of Products, Sums, and Scalar Multiples

For the next questions, let

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

1. Verify that $|AB| = |A||B|$.

   Solution:

   Clearly, $|A| = 1 \cdot 1 - 1 \cdot (-1) = 2$, and $|B| = 1 \cdot 2 - 0 \cdot 0 = 2$. Multiplying,

   $$|AB| = \left| \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right| = \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} = 2 \cdot 1 - 2 \cdot (-1) = 4$$

   which is clearly equal to $|A||B| = 2 \cdot 2 = 4$, as required.

2. Is $|A + B| = |A| + |B|$?

   Solution:

   Calculating,

   $$|A + B| = \left| \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right| = \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} = 2 \cdot 3 - 1 \cdot (-1) = 7$$

   which is clearly $\boxed{\text{not equal}}$ to $|A| + |B| = 4$.

3. Is $|2A| = 2|A|$? Calculating,

   $$|2A| = \begin{vmatrix} 2 & 2 \\ -2 & 2 \end{vmatrix} = 2 \cdot 2 - 2 \cdot (-2) = 8$$

   which is clearly $\boxed{\text{not equal}}$ to $2|A| = 4$.

4. For an $n \times n$ matrix A, what do you think $|2A|$ actually is? (**Hint:** row operations!!)

   Solution:

   It's clear that $2A$ is A with each row multiplied by 2. Since each such row operation mutliplies the determinant by 2, and there are n rows, we see that

   $$\boxed{|2A| = 2^n |A|}$$