# Homework 10

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1. Let $A$ and $B$ be defined as follows:

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 5 & 10 & -11/3 \\ 0 & 0 & 1/5 & 21 \\ 0 & 0 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 1/4 & 0 & 0 \\ 7 & 6 & 2 & 0 \\ 8 & 9 & 7 & -1 \end{bmatrix}$$

   Calculate $\left|A^{-1}B^{-1}\right|$.

   **Hint:** There's a good way and a bad way to do this question, and the bad way is much more painful!!

2. Prove that if 0 is an eigenvalue of $A$, then $A\vec{x} = \vec{0}$ has infinitely many solutions.

3. Assume that $A = PDP^{-1}$ where $P$ is some nonsingular matrix, and $D$ is a diagonal matrix with entries $d_1, d_2, \ldots, d_n$ along the diagonal. That is,

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

   Let $\vec{v}_j$ be the $j$th column of $P$. Then, prove that $\vec{v}_j$ is an eigenvector of $A$ with eigenvalue $d_j$. (This reinforces the connection between eigenvectors and showing that $A$ is similar to a diagonal matrix.)

   **Hint:** Do some algebraic manipulation with the equation $A = PDP^{-1}$ to make the question manageable! Also, you don't have to use this, but it'll make it a little tidier: what was our earlier formula for the $j$th column of a matrix using matrix multiplication?

4. Let $A$ be an $m \times n$ matrix with rows $\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_m$. Show that if $\vec{x}$ is orthogonal to $\vec{r}_i$ for all $i$, then $A\vec{x} = \vec{0}$.

5. Let $A$ be the following matrix:

$$A = \begin{bmatrix} 1 & 0 & a \\ 2 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Note that one of the entries of $A$ is the variable $a$. For which values of $a$ does the equation

$$A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

have infinitely many solutions? For which values of $a$ does it have no solutions?

**Hint:** What is the handy function that tells us whether $A$ is singular? How does that help?

6. Assume that $A = PDP^{-1}$ where $P$ is some nonsingular matrix, and $D$ is a diagonal matrix with entries $d_1, d_2, \ldots, d_n$ along the diagonal. (This is exactly the set up from Question 2.)

   (a) Use induction to show that for any positive integer $k$,

   $$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

   (b) Show that

   $$A^k = PD^kP^{-1}$$

   for any positive integer $k$.

   (c) Use parts (a) and (b) to calculate $A^7$, where

   $$A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$$

   **Hint:** You should probably start by diagonalizing $A$!

7. BONUS: Let $A$ be an $n \times n$ matrix. Recall that $\mathrm{tr}(A)$, the trace of $A$, is defined to be the sum of the diagonal entries of $A$: that is, if the entries of $A$ are $a_{ij}$, then

   $$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

   (a) As usual, let $p_A(x)$ be the characteristic polynomial of $A$. Use induction to show that $p_A(x)$ is a polynomial of degree $n$.

   (b) Now, write $p_A(x)$ as

   $$p_A(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$$

   That is, $c_0$ is the constant term in the polynomial, $c_1$ is the coefficient of $x$, and in general $c_i$ is the coefficient of $x^i$.

   i. Show that $c_n = 1$ for any $n \times n$ matrix $A$.

ii. Show that $c_{n-1} = \text{tr}(A)$ for any $n \times n$ matrix $A$.

(c) Now, say that $A$ has the $n$ eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, where the list includes repetitions: for example, if the characteristic polynomial was $(x-1)^2(x-2)$, we'd say that the eigenvalues are $1, 1, 2$. Use the results above to show that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \text{tr}(A)$$