Homework 7

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Note: You should use the contrapositive for some (but not all) of the proofs!

1. Find the inverses of the following matrices:

   (a) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

   (b) $\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$, if a_{11}, a_{22} are both non-zero.

   (c) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

   (d) $\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$, if $a_{11}, a_{22}, \ldots, a_{nn}$ are all non-zero.

2. Show that if $A\vec{x} = \vec{0}$ has a non-trivial solution, then A is singular.

3. (a) Give an example to show that $A + B$ can be singular if A and B are both nonsingular.

   (b) Give an example to show that $A + B$ can be nonsingular if A and B are both singular.

   Hint: 2×2 matrices should suffice for this one!

4. For the matrix A below, and for the values of (i, j) provided, calculate the (i, j) submatrix A_{ij} and the (i, j) minor $|A_{ij}|$.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 4 & 7 & -1 \\ -1 & 10 & 3 \end{bmatrix}$$

   (a) $(i, j) = (1, 1)$

   (b) $(i, j) = (2, 3)$

   (c) $(i, j) = (3, 1)$

5. Calculate the determinants of the following matrices, using the methods learned in class:

   (a) $\begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix}$.

   (b) $\begin{bmatrix} 0 & 1 & -1 \\ 2 & 4 & 5 \\ 3 & -1 & 1 \end{bmatrix}$.

   (c) $\begin{bmatrix} 1 & 0 & -2 & 0 \\ -1 & 3 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

   Hint: I recommend being judicious in the choice of row/column to expand along!

6. Show that if A has a row that's all 0s, then $|A| = 0$.

7. Prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

   (This kind of matrix is called a **Vandermonde matrix**.)

8. Show that if $|B| < 0$, then B can't be written as A^2 for any matrix A.

9. For the sets specified below, do the following:

   - Give an example of an element of the set
   - Check whether the element provided is in the set

   (a) $S = \{\vec{x} \mid \vec{x} \cdot [1,1,0] = 0\}$. Element to check: $\vec{v} = [1,1,1]$.

   (b) $S = \{A \mid A \text{ is a matrix of rank } 1\}$. Element to check: $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.

   (c) $S = \{c_1[1,1,1] + c_2[0,1,1] \mid c_1, c_2 \in \mathbb{R}\}$. Element to check: $\vec{v} = [1,2,1]$.