MATH 341 MIDTERM 1

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Show your work for all the problems. Good luck!

1. Let $\vec{v} = [1,2,3]$ and $\vec{w} = [1,1,1]$.

   (a) [5 pts] Calculate $3\vec{v} - 2\vec{w}$.

   Solution:

$$3\vec{v} - 2\vec{w} = 3[1,2,3] - 3[1,1,1] = [3,6,9] - [3,3,3] = \boxed{[0,3,6]}$$

   (b) [5 pts] Calculate $\cos(\theta)$ if θ is the angle between $\vec{v} - \vec{w}$ and $\vec{v} + \vec{w}$. Is θ acute or obtuse?

   Solution:

   Here, we have that

$$\begin{aligned} \vec{v} - \vec{w} &= [1,2,3] - [1,1,1] = [0,1,2] \\ \vec{v} + \vec{w} &= [1,2,3] + [1,1,1] = [2,3,4] \end{aligned}$$

   Using the formula for the cosine of an angle between two vectors:

$$\begin{aligned} \cos(\theta) &= \frac{(\vec{v} - \vec{w}) \cdot (\vec{v} + \vec{w})}{\|\vec{v} - \vec{w}\| \|\vec{v} + \vec{w}\|} = \frac{[0,1,2] \cdot [2,3,4]}{\|[0,1,2]\| \, \|[2,3,4]\|} \\ &= \frac{11}{\sqrt{0^2 + 1^2 + 2^2}\sqrt{2^2 + 3^2 + 4^2}} = \frac{11}{\sqrt{5}\sqrt{29}} \\ &= \boxed{\frac{11}{\sqrt{145}}} \end{aligned}$$

   An angle whose cosine is positive is $\boxed{\text{acute}}$.

   (c) [5 pts] Calculate $\text{proj}_{\vec{v}}\vec{w}$.

   Solution:

   We know that

$$\begin{aligned} \text{proj}_{\vec{v}}\vec{w} &= \left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|^2}\right)\vec{v} = \frac{[1,2,3] \cdot [1,1,1]}{1^2 + 2^2 + 3^2}[1,2,3] \\ &= \frac{6}{14}[1,2,3] = \boxed{\left(\frac{3}{7}, \frac{6}{7}, \frac{9}{7}\right)} \end{aligned}$$

2. Let A be an $n \times 2$ matrix with columns $\vec{v}_1$ and $\vec{v}_2$ in that order.

   (a) [5 pts] Find a vector
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
such that $A\vec{x} = 2\vec{v}_1 + 3\vec{v}_2$.

Solution:

Recall that
$$A\vec{x} = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2$$
Therefore, a simple way to make $A\vec{x} = 2\vec{v}_1 + 3\vec{v}_2$ is to let
$$\boxed{\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$

   (b) [5 pts] Show that if $\vec{w}$ is orthogonal to both $\vec{v}_1$ and $\vec{v}_2$ then $\vec{w}$ is orthogonal to $A\vec{x}$ for any vector $\vec{x}$. (Hint: linear combinations!)

Proof:
Assumptions: $\vec{w} \cdot \vec{v_1} = 0, \vec{w} \cdot \vec{v_2} = 0.$.
Need to show: $\vec{w} \cdot A\vec{x} = 0$ for all vectors $\vec{x}$.

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then, we have that
$$A\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2$$
Therefore,
$$\begin{aligned}\vec{w} \cdot A\vec{x} &= \vec{w} \cdot (x_1\vec{v}_1 + x_2\vec{v}_2) = \vec{w} \cdot (x_1\vec{v}_1) + \vec{w} \cdot (x_2\vec{v}_2) \\ &= x_1(\vec{w} \cdot \vec{v}_1) + x_2(\vec{w} \cdot \vec{x}_2) \\ &= x_1(0) + x_2(0) = 0\end{aligned}$$
using the assumption that $\vec{w} \cdot \vec{v}_1 = 0 = \vec{w} \cdot \vec{v}_2$. $\square$

3. Let A, B, C, D be the following matrices:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 \\ -3 & 3 \end{bmatrix}, C = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, D = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 2 \\ 3 & -2 & 4 \end{bmatrix}$$

Calculate the following matrix expressions, if possible. If it's not possible, justify why not.

(a) [5 pts] AC

Solution:

$$AC = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ 7 \end{bmatrix}}$$

(b) [5 pts] $CA + D$

Solution:

It's impossible: C is 3×1, while A is 2×3, and therefore they can't be multiplied. (To multiply two matrices, first matrix must be $m \times n$, while second matrix must be $n \times p$.)

a'

(c) [5 pts] $BA - 2A$

Solution:

$$\begin{aligned} BA - 2A &= \begin{bmatrix} -1 & 2 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 & 2 \\ -3 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \end{bmatrix} \\ &= \boxed{\begin{bmatrix} -3 & -1 & 2 \\ -3 & -2 & 1 \end{bmatrix}} \end{aligned}$$

(d) [5 pts] $D^2 + I_3 + D^T$

Solution:

$$\begin{aligned} D^2 + I_3 + D^T &= \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 2 \\ 3 & -2 & 4 \end{bmatrix}^2 + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 2 \\ 3 & -2 & 4 \end{bmatrix}^T \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 2 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 2 \\ 3 & -2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -2 \\ 0 & 2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -1 & -2 \\ 8 & -6 & 8 \\ 11 & -11 & 12 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 3 \\ -1 & 1 & -2 \\ 0 & 2 & 5 \end{bmatrix} \\ &= \boxed{\begin{bmatrix} 1 & 1 & 1 \\ 7 & -5 & 6 \\ 11 & -9 & 17 \end{bmatrix}} \end{aligned}$$

4. Find all solutions to the following systems of equations by first bringing the system into row-reduced echelon form. If there exists at least one solution, give an example and plug it back into the system to check that it works.

   (a) [5 pts]

$$\begin{aligned} x_1 + x_2 \ \ + x_3 \ &= 2 \\ 2x_1 + 3x_2 - x_3 \ &= 5 \\ x_1 + 2x_2 - 2x_3 &= -1 \end{aligned}$$

Solution:

Putting this system in augmented matrix form and row-reducing:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & -1 & 5 \\ 1 & 2 & -2 & -1 \end{array}\right] \xrightarrow{R_2:R_2-2R_1, R_3:R_3-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 1 & -3 & -3 \end{array}\right]$$

$$\xrightarrow{R_3:R_3-R_2, R_1:R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

This is now in row-reduced echelon form. Since the last equation corresponds to $0 = 4$, there are $\boxed{\text{no solutions.}}$

   (b) [5 pts]

$$\begin{aligned} 2x_1 - x_2 \qquad &= 1 \\ 5x_1 + x_2 + 3x_3 &= 5 \end{aligned}$$

Solution:

Putting this system in augmented matrix form and row-reducing:

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 5 & 1 & 3 & 5 \end{array}\right] \xrightarrow{R_2:R_2-2R_1} \left[\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 1 & 3 & 3 & 3 \end{array}\right]$$

$$\xrightarrow{R_1:R_1-2R_2,\text{ Swap } R_1, R_2} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 3 \\ 0 & -7 & -6 & -5 \end{array}\right]$$

$$\xrightarrow{R_2:R_2/(-7), R_1-3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3/7 & 6/7 \\ 0 & 1 & 6/7 & 5/7 \end{array}\right]$$

This corresponds to the system:

$$\begin{aligned} x_1 + \frac{3}{7}x_3 &= \frac{6}{7} \\ x_2 + \frac{6}{7}x_3 &= \frac{5}{7} \end{aligned}$$

Solving, all solutions of the system can be written as $[\frac{6}{7} - \frac{3}{7}x_3, \frac{5}{7} - \frac{6}{7}x_3, x_3]$. Letting $x_3 = 2$, a paticular solution is $[0, -1, 2]$. Plugging that into the system:

$$\begin{aligned} 2x_1 - x_2 \qquad &= 0 + 1 = 1 \\ 5x_1 + x_2 + 3x_3 &= 0 - 1 + 6 = 5 \end{aligned}$$

and therefore it works.

5. Let A and B be two matrices.

(a) [5 pts] Show that $AB = BA$ if and only if $(A+B)^2 = A^2 + 2AB + B^2$.

Solution:

Since this is an if and only if proof, we have to show two directions:

Proof:
Assumptions: $AB = BA$.
Need to show: $(A+B)^2 = A^2 + 2AB + B^2$.

Expanding out, and using the assumption that $AB = BA$:

$$\begin{aligned}(A+B)^2 &= (A+B)(A+B) = A^2 + AB + BA + B^2 = A^2 + AB + AB + B^2 \\ &= A^2 + 2AB + B^2\end{aligned}$$

concluding this direction.

Assumptions: $(A+B)^2 = A^2 + 2AB + B^2$.
Need to show: $AB = BA$.

By assumption, and using the same expansion as previously:

$$\begin{aligned}A^2 + 2AB + B^2 &= (A+B)^2 = A^2 + AB + BA + B^2 \\ &\Rightarrow AB = BA\end{aligned}$$

subtracting $A^2 + AB + B^2$ from both sides. This concludes the second direction, so we're done. □

(b) [5 pts] Recall that $\vec{e}_j$ is the vector with all 0 entries, except for a 1 in the jth place. Let A be an $m \times n$ matrix and let $\vec{e}_j$ be in $\mathbb{R}^n$. Prove that $A\vec{e}_j$ is the jth column of A.

Proof:
Let the (i, j) entry of A be a_{ij}. Since $A\vec{e}_j$ is an $m \times n$ matrix multiplied with an $n \times 1$ matrix, the result is an $m \times 1$ matrix: that is, a vector in R^m. Therefore,

$$\begin{aligned}\text{Entry } i \text{ of } A\vec{e}_j &= [\text{row } i \text{ of } A] \cdot \vec{e}_j \\ &= [a_{i1}, a_{i1}, \ldots, a_{in}] \cdot [0, \ldots, 1, \ldots, 0]\end{aligned}$$

where the 1 in the vector $[0, \ldots, 1, \ldots, 0]$ is in the jth place. Therefore,

$$\text{Entry } i \text{ of } A\vec{e}_j = a_{i1} \cdot 0 + a_{i2} \cdot 0 + \cdots + a_{ij} \cdot 1 + \cdots + a_{in} \cdot 0 = a_{ij}$$

and thus,

$$A\vec{e}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} = \text{Column } j \text{ of } A$$

□

6. BONUS: Let A be an $m \times n$ matrix with columns $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, and let B be A after some row operation, with columns $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$. For example, we could have

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 3 & -5 \end{bmatrix} \xrightarrow{R_2 : R_2 - R_1} \begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & -3 \end{bmatrix} = B$$

(a) [5 pts] If we have that $\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_n = \vec{0}$, show that

$$\vec{w}_1 + \vec{w}_2 + \cdots + \vec{w}_n = \vec{0}$$

Note: You can see that this works in the example above!

Solution:
Let the (i, j) entry of A be a_{ij}. Note that the columns of a matrix add up to $\vec{0}$ if and only if the sum of the entries in each row is 0. Therefore, the equation $\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_n = \vec{0}$ corresponds to

$$a_{i1} + a_{i2} + \cdots + a_{in} = 0 \text{ for all } i$$

There are three different types of row operations to consider. Let's do each case separately, showing that in each case the sum of the entries in each row after the row operstion is 0.

Swap R_i and R_j: In this case, if $k \neq i, j$ then the sum of the entries in row k hasn't changed, and it still 0. Because of the swap,

$$\text{Sum of the entries in row } i \text{ in } B = a_{j1} + a_{j2} + \cdots + a_{jn} = 0$$
$$\text{Sum of the entries in row } j \text{ in } B = a_{i1} + a_{i2} + \cdots + a_{in} = 0$$

Therefore, the sum of the entries in each row is still 0, as required.

Change R_i to cR_i: In this case, the only row that changes is row i, which becomes $[ca_{i1}, \ldots, ca_{in}]$. Therefore, we only need to show that the sum of the entries in row i is 0.

$$\begin{aligned} \text{Sum of the entries in row } i \text{ in } B &= ca_{i1} + ca_{i2} + \cdots + ca_{in} \\ &= c(a_{i1} + a_{i2} + \cdots + a_{in}) = 0 \end{aligned}$$

The sum of the entries in each row is still 0 yet again.

Chamge R_i to $R_i - cR_j$: Again, the only row that changes is row i, and it becomes $[a_{i1} - ca_{j1}, a_{i2} - ca_{j2}, \ldots, a_{in} - ca_{jn}]$. Therefore,

$$\begin{aligned} \text{Sum of the entries in row } i \text{ in } B &= (a_{i1} - ca_{j1}) + (a_{i2} - ca_{j2}) + \cdots + (a_{in} - ca_{jn}) \\ &= (a_{i1} + a_{i2} + \cdots a_{in}) - c(a_{j1} + a_{j2} + \cdots a_{jn}) = 0 \end{aligned}$$

Once again, the sum of the entries in each row is 0, so we're done.

(b) [2 pts] If instead

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$$

what relationship would there be between $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$? (You can just state it – no need to prove it!)

Solution:

Analogously to above, the relationship is:

$$\boxed{c_1\vec{w}_1 + \cdots + c_n\vec{w}_n = \vec{0}}$$