MATH 341 MIDTERM 2

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Show your work for all the problems. Good luck!

(1) Let A and B be defined as follows:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 3 & 4 \end{bmatrix}$$

(a) [5 pts] Demonstrate that A and B are row equivalent by providing a sequence of row operations leading from A to B.

Solution:

As usual, a good way to do this is to row reduce both A and B. Since they are row equivalent, they will have the same row reduced echelon form, which can be used to solve the question. Accordingly, we row reduce A:

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow{R_2:R_2-R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1:R_1-R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Now, we row reduce B:

$$\begin{bmatrix} 0 & 2 & 2 \\ 1 & 3 & 4 \end{bmatrix} \xrightarrow{\text{Swap } R_1 \text{ and } R_2} \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{R_2:\frac{1}{2}\times R_2} \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1:R_1-3R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

To get from A to B, perform the operations getting A to the common row-reduced echelon form, then reverse the operations that went into row reducing B. Therefore, a possible sequence of row operations is:

$$\boxed{R_2 \to R_2 - R_1, R_1 \to R_1 - R_2, R_1 \to R_1 + 3R_2, R_2 \to 2 \times R_2, \text{Swap } R_1 \text{ and } R_2}$$

(b) [5 pts] Check whether $[1, 4, 5]$ is in the row space of A and if it is, write it as a linear combination of the rows.

Solution:

A vector is in the row space of a matrix if it can be written a linear combination of the rows. Therefore, we need to solve for c_1 and c_2 such that

$$[1, 4, 5] = c_1[1, 1, 2] + c_2[1, 2, 3]$$

This simplifies to

$$[1, 4, 5] = [c_1 + c_2, c_1 + 2c_2, 2c_1 + 3c_2]$$

Writing this down as an augmented system and solving, we get

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & 5 \end{array}\right] \xrightarrow{R_2:R_2-R_1, R_3:R_3-2R_2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{array}\right] \xrightarrow{R_1:R_1-R_2, R_3:R_3-R_2} \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

This corresponds to $c_1 = -2, c_2 = 3$. Therefore, $[1, 4, 5]$ is in the row space of A, and

$$\boxed{[1, 4, 5] = -2[1, 1, 2] + 3[1, 2, 3]}$$

(2) Let A be an $m \times n$ matrix with rows $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m$. Let R_1 be the row operation Row $1 \to 2 \times$Row1, and R_2 be the row operation Swap Rows 1 and 2.

(a) [5 pts] What is the first row of $R_1(R_2(A))$ in terms of $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m$? (**Hint:** Think a little before deciding which row operations to do first!)

Solution:

We know that

$$A = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_m \end{bmatrix}$$

Therefore,

$$R_2(A) = \begin{bmatrix} \vec{r}_2 \\ \vec{r}_1 \\ \vdots \\ \vec{r}_m \end{bmatrix} \text{ and so } R_1(R_2(A)) = R_1\left(\begin{bmatrix} \vec{r}_2 \\ \vec{r}_1 \\ \vdots \\ \vec{r}_m \end{bmatrix}\right) = \begin{bmatrix} 2\vec{r}_2 \\ \vec{r}_1 \\ \vdots \\ \vec{r}_m \end{bmatrix}$$

Thus, the first row of $R_1(R_2(A))$ is $\boxed{2\vec{r}_2}$.

(b) [5 pts] What is the first row of $R_2(R_1(A))$ in terms of $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m$?

Solution:

Using the expression for A from part (a), we see that

$$R_1(A) = \begin{bmatrix} 2\vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_m \end{bmatrix} \text{ and so } R_2(R_1(A)) = R_2\left(\begin{bmatrix} 2\vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_m \end{bmatrix}\right) = \begin{bmatrix} \vec{r}_2 \\ 2\vec{r}_1 \\ \vdots \\ \vec{r}_m \end{bmatrix}$$

Thus, the first row of $R_2(R_1(A))$ is $\boxed{2\vec{r}_2}$.

(c) [5 pts] Prove that if $\vec{x}$ and $\vec{y}$ are both in the row space of A, then so is $\vec{x} + \vec{y}$.

Proof:
Assumptions: $\vec{x}$ and $\vec{y}$ are both in the row space of A .
Need to show: $\vec{x} + \vec{y}$ is in the row space of A.

Since $\vec{x}$ is in the row space of A, it is a linear combination of the rows of A. Since the rows of A are $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$, here exist constants $c_1, c_2, \cdots, c_n$ such that

$$\vec{x} = c_1\vec{r}_1 + \cdots + c_n\vec{r}_n$$

Similarly, since $\vec{y}$ is in the row space of A, there exist constants $d_1, d_2, \cdots, d_n$ such that

$$\vec{y} = d_1\vec{r}_1 + \cdots + d_n\vec{r}_n$$

Therefore,

$$\begin{aligned} \vec{x} + \vec{y} &= (c_1\vec{r}_1 + \cdots + c_n\vec{r}_n) + (d_1\vec{r}_1 + \cdots + d_n\vec{r}_n) \\ &= (c_1 + d_1)\vec{r}_1 + \cdots + (c_n + d_n)\vec{r}_n \end{aligned}$$

Since $c_1 + d_1, c_2 + d_2, \ldots, c_n + d_n$ are clearly all constants, the above expression shows that $\vec{x} + \vec{y}$ is in the row space of A, as required. $\square$

(3) Let A be defined as below:

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix}$$

(a) [5 pts] Calculate A^{-1} if A is nonsingular, or prove that it is singular.

Solution:

As usual, we augment A with the identity matrix and row reduce A: if A row reduces to the identity matrix, then the final result is $[I_n|A^{-1}]$, and otherwise A is singular.

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2:R_2+2R_1} \left[\begin{array}{ccc|ccc} -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3:R_3-R_1} \left[\begin{array}{ccc|ccc} -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1:(-1)\times R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3:R_3-R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & -1 & -3 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1:R_1-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & -1 & -3 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1:R_2+2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & -4 & -1 & 2 \\ 0 & 0 & -1 & -3 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3:(-1)\times R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & -4 & -1 & 2 \\ 0 & 0 & 1 & 3 & 1 & -1 \end{array}\right]$$

Therefore, A is nonsingular, and

$$\boxed{A^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ -4 & -1 & 2 \\ 3 & 1 & -1 \end{bmatrix}}$$

(b) [5 pts] Calculate $|A|$ by using row or column expansion.

Solution:

Expanding along the first row, we have that

$$\begin{vmatrix} -1 & 0 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{vmatrix} = (-1) \cdot \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + 0 \cdot \begin{vmatrix} 2 & 0 \\ -1 & 2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix}$$
$$= -2 + 3 = \boxed{1}$$

(c) [5 pts] Calculate $|A|$ using row reduction (feel free to reuse your work from part (a) for this!)

Solution:

From part (a), we know the sequence of row reductions that gets A into row reduced row echelon form. Tracking what those row reductions do to the determinant, we get that

$$|A| \xrightarrow{R_2:R_2+2R_1} |A| \xrightarrow{R_3:R_3-R_1} |A| \xrightarrow{R_1:(-1)\times R_1} -|A| \xrightarrow{R_3:R_3-R_2} -|A|$$
$$\xrightarrow{R_1:R_1-R_3} -|A| \xrightarrow{R_2:R_2+2R_3} -|A| \xrightarrow{R_3:(-1)\times R_3} |A|$$

Therefore, the determinant of the row reduced echelon form of A is precisely $|A|$, and hence

$$|A| = \left| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = \boxed{1}$$

Note that we got the same answer as in part (b), as expected!

(4) Define A to be the following matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

(a) [5 pts] Find the characteristic polynomial of A.

Solution:

By definition, the characteristic polynomial of A is

$$\begin{aligned} p_A(x) = |xI_n - A| &= \left| x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} x-1 & -2 \\ 0 & x-1 \end{vmatrix} \\ &= (x-1)(x-1) = \boxed{(x-1)^2} \end{aligned}$$

(b) [5 pts] Find the eigenvalues of A.

Solution:

The eigenvalues of A are the roots of the characteristic polynomial of A. Therefore, setting $p_A(x)$ to 0, we get

$$0 = (x-1)^2 \ \Rightarrow \ x = 1$$

Therefore, the only eigenvalue of A is $\boxed{\lambda = 1}$.

(c) [5 pts] Pick an eigenavalue of A, and find the fundamental eigevenctors for that eigenvalue.

Solution:

A only has the one eigenvalue 1, therefore we need to find the fundamental eigenvector for it. To find that, solve the system $(\lambda I - A)\vec{x} = \vec{0}$. Since $\lambda = 1$,

$$\lambda I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

Thus, the system $(\lambda I - A)\vec{x} = \vec{0}$ corresponds to the following augmented matrix (which row reduces very simply):

$$\left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1:1/2\times R_1} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

This system clearly corresponds to

$$\begin{aligned} c_1 &= c_1 \\ c_2 &= 0 \end{aligned}$$

Therefore,

$$E_1 = \{\vec{x} \mid A\vec{x} = \vec{x}\} = \left\{ \begin{bmatrix} c_1 \\ 0 \end{bmatrix} \;\middle|\; c_1 \in \mathbb{R} \right\} = \left\{ c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \;\middle|\; c_1 \in \mathbb{R} \right\}$$

Thus, the fundamental eigenvector is $\boxed{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$. (Of course, any scalar multiple of this is also correct!)

(5) Define the *permanent* $\text{per}(A)$ of a matrix A very similarly to the determinant: for a 1×1 matrix $A = [a_{11}]$, $\text{per}(A) = a_{11}$, and for an $n \times n$ matrix it is defined recursively as

$$\text{per}(A) = a_{11}\text{per}(A_{11}) + a_{12}\text{per}(A_{12}) + \cdots + a_{1n}\text{per}(A_{1n})$$

where A_{ij} is defined as usual to be the matrix A with row i and column j crossed out. As you can see, we define it using expansion along the first row, except that the sum doesn't alternate the way it does with determinants.

For example,

$$\text{per}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = 1 \cdot \text{per}([3]) + 2 \cdot \text{per}([2]) = 1 \cdot 3 + 2 \cdot 2 = 7$$

(a) [5 pts] Calculate the permanent of the following matrix:

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$$

Solution:

$$\text{per}\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} = 1 \cdot \text{per}([4]) + 2 \cdot \text{per}([-1]) = 1 \cdot 4 + 2 \cdot (-1) = \boxed{2}$$

(b) [5 pts] Prove that if $\text{per}(A) \neq 0$, then at least one entry of the first row of A is nonzero.

Proof:
Let's use the contrapositive. The contrapositive of "C implies D" is "not D implies not C." Our original statement translates to: "$\text{per}(A) \neq 0$ implies that at least one entry of the first row of A is nonzero." Therefore, the contrapositive is "All of the entries of the first row of A being 0 implies that $\text{per}(A) = 0$."

Assume: The first row of A is 0.
Need to show: $\text{per}(A) = 0$.

By definition,

$$\begin{aligned} \text{per}(A) &= a_{11}\text{per}(A_{11}) + a_{12}\text{per}(A_{12}) + \cdots + a_{1n}\text{per}(A_{1n}) \\ &= 0 \cdot \text{per}(A_{11}) + 0 \cdot \text{per}(A_{12}) + \cdots + 0 \cdot \text{per}(A_{1n}) = 0 \end{aligned}$$

as required. □

(c) [5 pts] Prove that if A is an $n \times n$ matrix all of whose entries are positive, then $\text{per}(A)$ is positive as well. (This is NOT true for the determinant, by the way!)

Proof:
This proof uses induction on n. The nth statement is "For any $n \times n$ matrix A all of whose entries are positive, $\text{per}(A)$ is positive as well."

Base case:
Show that the statement holds for $n = 1$.

Asssume: A is a 1×1 matrix with positive entries.
Need to show: $\text{per}(A) > 0$.

Let A be a 1×1 matrix. Then, we can say that $A = [a_{11}]$. In that case,

$$\text{per}(A) = a_{11} > 0$$

as required.

Inductive step:
Here, we show that the statement for $n = k$ implies the statement for $n = k + 1$.

Asssume: If A is a $k \times k$ matrix with positive entries, then $\text{per}(K) > 0$.
Need to show: If A is a $(k + 1) \times (k + 1)$ matrix with positive entries, then $\text{per}(K) > 0$.

Let A be a $(k + 1) \times (k + 1)$ matrix with entries a_{ij}. By definition,

$$\text{per}(A) = a_{11}\text{per}(A_{11}) + a_{12}\text{per}(A_{12}) + \cdots + a_{1(k+1)}\text{per}(A_{1(k+1)})$$

By the inductive hypothesis, since A_{ij} is a $k \times k$ matrix, $\text{per}(A_{ij}) > 0$ for each i and j. Furthermore, we know that $a_{ij} > 0$ for each i and j. Since the product of a pair of positive numbers is positive, this shows that

$$a_{11}\text{per}(A_{11}) > 0, a_{12}\text{per}(A_{12}) > 0, \ldots, a_{1(k+1)}\text{per}(A_{1(k+1)}) > 0$$

Adding up $k + 1$ positive numbers clearly leads to a positive number, and thus we get that $\text{per}(A) > 0$, as required.

□

(6) Consider the set $S_A = \{\vec{x} \mid A\vec{x} = [1,1]^T\}$

(a) [5 pts] Let A be defined as

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Without solving any linear systems, check whether $\vec{x} = [1,-1,1]$ is in S_A.

Solution:

By definition, $\vec{x}$ is in S_A if

$$A\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Therefore, we just need to check whether $A\vec{x}$ is correct. Checking,

$$A\vec{x} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Therefore, $\boxed{\vec{x} \text{ is not in } S_A.}$

(b) [5 pts] Now let A be some $2 \times n$ matrix (not necessarily the matrix in part (a), although it is some fixed matrix.) If the set S_A contains infinitely many vectors, what does that tell you about the set of solutions to the system $A\vec{x} = \vec{0}$?

Solution:

From equivalences we learned earlier in the course, the fact that S_A contains infinitely many vectors means precisely that $\boxed{\text{the set of solutions to } A\vec{x} = \vec{0} \text{ is infinite.}}$

(7) BONUS: Row operations can actually be thought of as matrix multiplication on the left: to be precise, for every row operation R, there exists a matrix M_R such that $M_R A = R(A)$. In this question, we will explore how that works.

(a) [2 pts] Let

$$H = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If A is a $3 \times n$ matrix, then HA is equal to $R(A)$ for some row operation R. What is that row operation? You don't need to prove it. (**Hint:** Try some examples!)

Solution:

To try an example, let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Then,

$$HA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

Clearly, HA is A with rows 1 and 2 swapped. Therefore,

$$\boxed{R = \text{Swap Rows 1 and 2}}$$

(b) [4 pts] Find a $n \times m$ matrix G such that for every $n \times n$ matrix A, $GA = R(A)$, where the row operation R is Row $i \to$ Row $i + c \cdot$ Row j. (You don't need to prove that it works!)

Solution:

We define the matrix G entry by entry. If the (k, l) entry of G is g_{kl} (we're not using i and j since those letters were already used for something specific), then we have that

$$g_{kl} = \begin{cases} 1 & k = l \\ c & (k, l) = (i, j) \\ 0 & \text{otherwise} \end{cases}$$

If this is too much notation, what the above description says is that G is equal to I_n everywhere except at the (i, j) entry, and that $g_{ij} = c$.

As an example, say that we want a 3×3 matrix G that performs the row operation $R =$ Row $2 \to$ Row $2 + 5 \cdot$ Row 1. According to the above description, G will be precisely I_3, except that $g_{21} = 5$. Thus,

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let's check that this works. Define

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 3 & 4 & 5 \end{bmatrix}$$

Then, we have that

$$GA = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 6 & -4 & 1 \\ 3 & 4 & 5 \end{bmatrix}$$

which is precisely $R(A)$, as expected.

(c) [4 pts] Prove that your answer from part (b) works.

Solution:

Let G be defined as above. We need to show that for any $n \times m$ matrix A, $GA = R(A)$, where R is the row operation Row $i \to$ Row $i + c \cdot$ Row j. We'll work this out entry by entry. Let the (k,l) entry of A be a_{kl} (we don't use the letters i and j because they've been used already, like before.) Then,

$$R(A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} + ca_{j1} & a_{i2} + ca_{j2} & \cdots & a_{in} + ca_{jn} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

To be precise, we have that

$$(k,l) \text{ entry of } R(A) = \begin{cases} a_{kl} & \text{if } k \neq i \\ a_{il} + ca_{jl} & \text{if } k = i \end{cases}$$

Let us now show that the (k,l) entry of GA is the same. As usual,

$$(k,l) \text{ entry of } GA = (\text{row } k \text{ of G}) \cdot (\text{column } l \text{ of A})$$

If $k \neq i$, then it's clear that the kth row of G is just $[0, 0, \ldots, 1, \ldots, 0]$ where the 1 is in the kth place. Therefore, if $k \neq i$,

$$(k,l) \text{ entry of } GA = [0, \ldots, 1, \ldots, 0] \cdot [a_{1l}, a_{2l}, \ldots, a_{nl}] = a_{kl}$$

If $k = i$, then the kth row of G is the ith row of G, which is all 0s except a 1 in the ith place and a c in the jth place. Therefore,

$$(i,l) \text{ entry of } GA = [0, \ldots, 1, \ldots, c, \ldots, 0] \cdot [a_{1l}, a_{2l}, \ldots, a_{nl}] = a_{il} + ca_{jl}$$

Therefore, we see that

$$(k,l) \text{ entry of } GA = \begin{cases} a_{kl} & \text{if } k \neq i \\ a_{il} + ca_{jl} & \text{if } k = i \end{cases}$$

which is precisely what we got for the (k,l) entry of $R(A)$. Therefore, $R(A) = GA$, as required.