Practice Final for Math 341

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1. Define the following vectors and matrices:

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{y} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, C = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

   For each of the following questions, either calculate the quantity in the question or explain why it's impossible.

   (a) The unit vector in the direction $\vec{x} + \vec{y}$.

   (b) $\text{proj}_{\vec{y}}\vec{x}$.

   (c) $A\vec{x} + B\vec{y}$.

   (d) $A + 2AC$.

   (e) The angle θ between $\vec{x}$ and $\vec{y}$.

   (f) AB

2. Let A be an $m \times 3$ matrix with columns $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

   (a) Let B be the following matrix:

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 0 \end{bmatrix}$$

   Write the second column of AB in terms of $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$.

   (b) Prove that if $A^T\vec{x} = \vec{0}$, then $\vec{x}$ is orthogonal to $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$.

3. Prove that if $AB = BA$, then $A^nB = BA^n$ for all positive integer n.

4. Find all solutions to the following systems of equations. For each system, state how many solutions it has.

   (a)

$$\begin{aligned} x_1 + x_2 \;\; + x_3 &= 1 \\ x_1 + 2x_2 - 5x_3 &= -2 \end{aligned}$$

(b)

$$\begin{aligned} a + b &= 0 \\ a + 3b &= 0 \end{aligned}$$

5. Recall that if R is a row operation, then $R(AB) = R(A)B$. Now, let C be the column operation Column $1 \to 2 \times$ Column 1. Prove or disprove: $C(AB) = C(A)B$ for any matrices A and B (such that AB is defined.)

6. Let A be the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

   (a) What is the rank of A?

   (b) What is the determinant of A? (Hint: this is easiest using the answer from (a).)

   (c) Does A have an inverse? If so, calculate it; if not, explain why not.

   (d) Let $\vec{b}$ be a vector such that the system $A\vec{x} = \vec{b}$ has the solution $\vec{x}_0$. How many solutions does the system $A\vec{x} = \vec{b}$ have overall?

7. Let A be defined as follows:

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

   (a) Calculate A^{-1}.

   (b) Calculate $|A|$ using row or column expansion.

8. (a) Find a counterexample to the following statement: If A is an $m \times n$ amtrix, and B is an $n \times m$ matrix, then $AB = I_m$ implies $BA = I_n$.

   (b) If $A^2 = A$, what are the possible values of $|A|$?

   (c) Find the determinant of the following matrix:

$$A = \begin{bmatrix} 7 & 2026 & 1251 \\ 7 & 2025 & -900 \\ 7 & 2026 & 1351 \end{bmatrix}$$

   Hint: Don't use row or column expansion!

9. Diagonalize the following matrices A, or show that it is impossible: (That is, we're looking for a diagonal matrix D and a matrix P such that $P^{-1}AP = D$, or trying to show they can't be found.)

(a)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

10. Check whether the following set is a vector space: V is all matrices of the form

$$A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

and the operations are defined as:

$$A \oplus B = AB$$

$$c \odot \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & ac \\ 0 & 1 \end{bmatrix}$$

11. Answer the following questions:

(a) Is $W = \{[x, y] \mid x \geq 0, y \geq 0\}$ a subspace of $\mathbb{R}^2$?

(b) If A is an $m \times n$ matrix, is

$$W = \{\vec{x} \mid A\vec{x} = \vec{0}\}$$

a subspace of $\mathbb{R}^n$?

(c) Is $W = \{[x, y, 0] \mid x, y \in \mathbb{R}\}$ a subspace of $\mathbb{R}^3$?

(d) Is $W = \{A \in \mathcal{M}_{22} \mid A \text{ is singular}\}$ a subspace of $\mathcal{M}_{22}$?

12. Check whether the following sets S are linearly dependent or independent. If they are linearly dependent, write down a linear combination of the elements in S that's equal to $\vec{0}$.

(a)

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

(b)

$$S = \{[1, -1, 1], [0, 1, 3], [2, 1, 11]\}$$

13. Prove that if $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a linearly independent subset of V, then for any $\vec{x}$ in V, there is at most one way to write it as a linear combination of the elements in S.

14. Check whether the following are bases of the given vector spaces. You may use the fact that the dimension of $\mathbb{R}^n$ is n.

(a) Is $\{[1,1],[0,1],[2,3]\}$ a basis of $\mathbb{R}^2$?

(b) Is $\{[1,1,1],[1,2,3],[3,4,5]\}$ a basis of $\mathbb{R}^3$?

(c) If $\{[3,4],[-1,2]\}$ a basis of $\mathbb{R}^2$?

15. If we're given that a vector space V contains a set S of size 21 such that S is linearly independent, what does that tell us about the dimension of V?

16. Check whether the following functions $T : V \to W$ are linear transormations. If they are, prove it; if they are not, provide a counterexample.

(a) $T : \mathbb{R}^3 \to \mathbb{R}^2$,

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - y + z \\ 2x - z \end{bmatrix}$$

(b) $T : \mathcal{M}_{22} \to \mathbb{R}^2$,

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ab \\ cd \end{bmatrix}$$

(c) Let V be a vector space, and let c be a fixed scalar. Then, $T : V \to V$, given by

$$T(\vec{v}) = c\vec{v}$$

17. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation satisfying

$$T \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, T \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Find the matrix A such that $T(\vec{x}) = A\vec{x}$.