Midterm 2: Concepts to Review

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The second midterm will cover Sections 2.1, 2.2, 2.3, 2.4, 3.1, 3.2, 3.3, and 3.4, with the usual caveat that you only need to know the things covered in lecture (there are a number of concepts we've skipped.) I will not *explicitly* test the material from the first midterm, but since that material has been continuously utilized since, you're expected to know it.

1. Solving Systems of Equations (Sections 2.1 and 2.2) – we did some of this on the first midterm, but there were a number of things we've skipped. Of course, you'll be expected to know how to solve linear systems on the exam, too!

   - What the system $A\vec{x} = \vec{b}$ corresponds to as an augmented system of equations: that is, $[A|\vec{b}]$.
   - If R is a row operation, then $R(AB) = R(A)B$.
   - What a homogoneous system is: that is, the system $A\vec{x} = \vec{0}$.

2. Equivalent Systems, Rank, and Row Space (Section 2.3)

   - Two systems are *equivalent* if they have exactly the same solutions.
   - A matrix C is *row equivalent* to a matrix D if C is obtained from C with a finite number of row operations.
   - Knowing how to 'reverse' row operations: this can be used to show that if C is row equivalent to D, then D is row equivalent to C.
   - How to test if two matrices are row equivalent: two matrices are row equivalent if they have the exact same reduced row echelon form
   - The definition of the *rank* of A:

     $$\begin{aligned}\text{rank}(A) &= \{\text{number of non-zero rows in the rref of } A\}\\ &= \{\text{number of pivotal columns in the rref of } A\}\end{aligned}$$

   - When a homogeneous system (that is, the system $A\vec{x} = \vec{0}$) has one or infinitely many solutions:
     (a) If $\text{rank}(A) < n$, then the system has a non-trivial solution
     (b) If $\text{rank}(A) = n$, then the system has the one solution $\vec{x} = \vec{0}$.

- The definition of the *row space* of a matrix, and how to check whether a vector is in the row space: the row space of A is the set of linear combinations of the rows of A.
- If A is row equivalent to B, then the row space of A is equal to the row space of B.

3. Inverses of Matrices (Section 2.4)

   - For a square $n \times n$ matrix A, B is the *inverse* of A if
   $$AB = BA = I_n$$
   This is denoted by $B = A^{-1}$.
   - If A and B are $n \times n$ matrices such that $AB = I_n$, then we have that A and B are inverses: that is, $BA = I_n$.
   - A square matrix is *singular* if and only if it does not have an inverse. If it does have an inverse, then it is *nonsingular*.
   - Inverse properties: Let A and B be nonsingular $n \times n$ matrices. Then,
     (a) A^{-1} is nonsingular and $(A^{-1})^{-1} = A$.
     (b) AB is nonsingular, and $(AB)^{-1} = B^{-1}A^{-1}$
     (c) A^T is nonsingular and $(A^T)^{-1} = (A^{-1})^T$.
   - For a 2×2 matrix A,
   $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
   (In fact, you now know that the $\frac{1}{ad-bc}$ term corresponds to $\frac{1}{|A|}$!)
   - How to find the inverse of any matrix:
     (a) If A is $n \times n$, then augment A with I_n, getting the system $[A|I_n]$.
     (b) Row reduce A, performing the same operations on both sides of the augmenting bar
     (c) If A row reduces to I_n, then it has an inverse, and the end result is $[I_n|A^{-1}]$.
     (d) If the row reduced echelon form of A is not I_n, then A has no inverse: it is singular.
   - An $n \times n$ matrix A is nonsingular if and only if $\text{rank}(A) = n$.
   - The system $A\vec{x} = \vec{b}$ has a unique solution if and only if A is nonsingular. If A is singular, then it either has no solutions or infinitely many solutions.

4. Determinants (Sections 3.1, 3.2, and 3.3 – we did it out of order!)

   - For a 2×2 matrix, the determinant is calculated as follows:
   $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- For bigger matrices, the determinant is calculated using row or column expansion. Before defining this, define A_{ij} to be the matrix we get by crossing out row i and column j of A. Then,

$$\mathcal{A}_{ij} = (-1)^{i+j}|A_{ij}|$$

  This is called the (i, j) cofactor of A, while $|A_{ij}|$ is the (i, j) minor of A.

- Calculating the determinant using row expansion: if A is an $n \times n$ matrix, then for any i,

$$\begin{aligned} |A| &= a_{i1}\mathcal{A}_{i1} + a_{i2}\mathcal{A}_{i2} + \cdots + a_{i1}\mathcal{A}_{i1} \\ &= (-1)^{i+1}a_{i1}\,|A_{i1}| + (-1)^{i+2}a_{i2}\,|A_{i1}| + \cdots + (-1)^{i+n}a_{in}\,|A_{in}| \end{aligned}$$

- Calculating the determinant using column expansion: if A is an $n \times n$ matrix, then for any j,

$$\begin{aligned} |A| &= a_{1j}\mathcal{A}_{1j} + a_{2j}\mathcal{A}_{i2} + \cdots + a_{nj}\mathcal{A}_{nj} \\ &= (-1)^{j+1}a_{1j}\,|A_{1j}| + (-1)^{j+2}a_{2j}\,|A_{1j}| + \cdots + (-1)^{j+n}a_{nj}\,|A_{nj}| \end{aligned}$$

5. Properties of Determinants (Sections 3.1, 3.2, and 3.3)

   - $|AB| = |A||B|$ for all $n \times n$ matrix A and B.
   - $|A^T| = |A|$.
   - However, it's not true that $|A + B| = |A| + |B|$ in general!
   - $|A| = 0$ if and only if A is singular – that is, doesn't have an inverse.
   - Effects of row operations on the determinant (summary posted as solutions of the in-class work.)
   - For an upper triangular $n \times n$ matrix A,

$$|A| = a_{11}a_{22}\cdots a_{nn}$$

   - Using row operations to calculate determinants
   - If A is an $n \times n$ matrix, then the following statements are all equivalent (all imply each other):
     (a) A is singular (doesn't have an inverse)
     (b) $\text{rank}(A) < n$
     (c) $|A| = 0$
     (d) $A\vec{x} = \vec{0}$ has a nontrivial solution.
     (e) $A\vec{x} = \vec{b}$ either has no solutions or infinitely many solutions (depending on $\vec{b}$.)
   - Similarly, if A is an $n \times n$ matrix, then the following statements are all equivalent (all imply each other):

(a) A is nonsingular (has an inverse)
(b) $\text{rank}(A) = n$
(c) $|A| \neq 0$
(d) $A\vec{x} = \vec{0}$ only has the trivial solution $\vec{x} = \vec{0}$.
(e) $A\vec{x} = \vec{b}$ has exactly one solution for all $\vec{b}$.

6. Eigenvalues and Eigenvectors (Section 3.4)

   - λ is an *eigenvalue* of A if there exists a nonzero $\vec{x}$ such that $A\vec{x} = \lambda x$. In that case, $\vec{x}$ is an *eigenvector* of A with eigenvalue λ.
   - The *characteristic polynomial* of A is defined to be

   $$p_A(x) = |xI_n - A|$$

   - To find eigenvalues, solve $0 = p_A(x)$.
   - To find the eigenvectors corresponding to λ, solve the system

   $$(\lambda I_n - A)\vec{x} = \vec{0}$$

7. New Proof Techniques (and of course, you're responsible for the kind of direct proofs we've been doing before, too.)

   - Proof by contrapositive: instead of proving that A implies B, prove that (not B) implies (not A).
   - Proof by induction: to show that some statement holds for any positive integer n (possibly for all n above a certain number), prove the following:
     (a) Base case: the statement holds for the smallest possible value of n, often $n = 1$ but not always.
     (b) Inductive step: if you assume that the statement holds for $n = k$, then it holds for $n = k + 1$.

8. Sets: using set notation, giving example of elements of sets and checking whether things are in sets.