Midterm 1: Concepts to Review

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The first midterm will cover Sections 1.1, 1.2, 1.4, 1.5, and what we've done of 2.1 and 2.2 – that is, row-reducing matrices and using the row-reduced echelon form to find all solutions of a set of equations.

1. Fundamental Operations with Vectors (Section 1.1)

   - Definition of a vector
   - Addition of vectors, multiplying vectors by a scalar.
   - The length $\|\vec{x}\|$ of a vector:

     $$\|[x_1, x_2, \ldots, x_n]\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

   - Unit vectors; finding the unit vector in the same direction as a vector $\vec{x}$.
   - Using the properties of addition and scalar multiplication (Theorem 1.3)
   - Definition of a linear combination: a vector $\vec{v}$ is a linear combination of vectors $\vec{v}_1, \ldots, \vec{v}_n$ if it's possible to find scalars $c_1, c_2, \ldots, c_n$ such that

     $$\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$$

2. The dot product (Section 1.5)

   - The definition of the dot product of $\vec{x} = [x_1, x_2, \ldots, x_n]$ and $\vec{y} = [y_1, y_2, \ldots, y_n]$:

     $$\begin{aligned}\vec{x} \cdot \vec{y} &= [x_1, x_2, \ldots, x_n] \cdot [y_1, y_2, \ldots, y_n] \\ &= x_1y_1 + x_2y_2 + \cdots + x_ny_n\end{aligned}$$

   - The Cauchy-Schwarz inequality: $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\|\|\vec{y}\|$.
   - The triangle inequality: $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$.
   - The angle between two vectors: if θ is the angle between $\vec{x}$ and $\vec{y}$, then θ is defined to be between 0 and π and it satisfies

     $$\vec{x} \cdot \vec{y} = \|\vec{x}\| \, \|\vec{y}\| \cos(\theta)$$

- Using the dot-product to check for orthogonality, being parallel, being in opposite directions
- The definition of a projection of $\vec{b}$ onto $\vec{a}$, denoted by $\text{proj}_{\vec{a}}\vec{b}$:

$$\text{proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2} \right) \vec{a}$$

3. Fundamental Operations with Matrices (Section 1.4)

- Definition of an $m \times n$ matrix
- Matrix addition, multiplying matrices by scalars
- Definitions of special matrices: square matrices, diagonal matrices, identity matrices, upper triangular matrices, lower triangular matrices, zero matrices
- Properties of addition and scalar multiplication of matrices (Theorem 1.11)
- The transpose of a matrix and its properties

4. Matrix Multiplication (Section 1.5)

- How to multiply matrices: if A is an $m \times n$ matrix and B is an $n \times p$ matrix, then AB is an $m \times p$ matrix such that

$$(i, j) \text{ entry of } AB = (\text{row } i \text{ of } A) \cdot (\text{column } j \text{ of } B)$$

  where the $\cdot$ means the dot product
- Matrix multiplication is **not** commutative: we rarely have $AB = BA$.
- The following identities:

$$\begin{aligned} k\text{th column of } AB &= A(k\text{th column of } B) \\ k\text{th row of } AB &= (k\text{th column of } A)B \end{aligned}$$

- Fundamental properties of matrix multiplication (Theorem 1.14)
- Powers of square matrices
- Matrix multiplication and transposes: $(AB)^T = B^T A^T$.
- Linear combinations from matrix multiplication: how to find a linear combination of the rows of A by multiplying A on the left by a row vector, how to find a linear combination of the columns of A by multiplying A on the right by a column vector

5. Systems of Equations (Sections 2.1 and 2.2, what we've covered of them.)

- Writing down a system of linear equations in augmented matrix form
- Getting the system into row-reduced echelon form

- Using the row-reduced echelon form to find all solutions to the system
- The criteria for when a system in row-reduced form has
  - No solutions
  - Exactly one solution
  - Infinitely many solutions

6. And last but definitely not least... PROOFS! Being able to do the kinds of relatively short proofs we've done on the homework.
   - Write down what you're starting from and what you're proving
   - Making sure to start from the assumption, and work towards the conclusion
   - Explain your steps as if you had to explain what's going on to a classmate