# Solutions to Systems of Equations from September 15th

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Find all solutions to the following systems of equations:

1.

$$\begin{aligned} x_1 + x_2 \qquad &= 2 \\ 2x_1 + 3x_2 + x_3 &= 5 \\ - x_2 \ - x_3 &= -1 \end{aligned}$$

**Solution:** Putting this in augmented matrix form, we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 2 & 3 & 1 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right]$$

We work on the columns from left to right. The leftmost unfinished column that contains the first non-zero entry of a row is column 1, since it contains the first non-zero entries of rows 1 and 2. It already has a 1 in it from the first row, so we will use that 1 to cancel out the other non-zero entries of this column:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 2 & 3 & 1 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right] \xrightarrow{R_2:R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{array}\right]$$

The first column is now done. The next column that contains the first non-zero entry of a row is column 2, which contains the first non-zero entries of both rows 2 and 3. Use the 1 in row 2 to cancel out the other non-zero entries of column 2:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{array}\right] \xrightarrow{R_3:R_3+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1:R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is now in row-reduced echelon form, since there are no more unfinished columns which contains the first non-zero entry of a row. Rewriting this as equations again, we see that they are:

$$\begin{aligned} x_1 \quad - x_3 &= 1 \\ x_2 + x_3 &= 1 \\ 0 &= 0 \end{aligned}$$

(Note that the last equation comes from $0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0$.) Now, the variables corresponding to pivots are the dependent variables, and the variables not corresponding to pivots are the independent variables. Therefore, the only independent variable is $x_3$, so we solve for everything in terms of $x_3$, getting:

$$\begin{aligned} x_1 &= 1 + x_3 \\ x_2 &= 1 - x_3 \\ x_3 &= x_3 \end{aligned}$$

Rewriting this in vector form, all solutions to the above system can be written as $[1 + x_3, 1 - x_3, x_3]$ for any value of $x_3$.

While this is not a necessary part of the solution, let's check that this solution works for any value of $x_3$ by plugging it back into the original equations:

$$\begin{aligned} (1 + x_3) + (1 - x_3) \qquad\quad &= 2 \\ 2(1 + x_3) + 3(1 - x_3) + x_3 &= 5 \\ -(1 - x_3) \quad - x_3 &= -1 \end{aligned}$$

so we see that all the $x_3$'s cancel out, and it works.

2.

$$\begin{aligned} 2x_1 + x_2 \; + 3x_3 \qquad\quad &= 16 \\ 3x_1 + 2x_2 \qquad\quad + x_4 &= 16 \\ 2x_1 \qquad + 12x_3 - 5x_4 &= 5 \end{aligned}$$

**Solution:** Rewriting this as an augmented matrix, we get

$$\left[\begin{array}{cccc|c} 2 & 1 & 3 & 0 & 16 \\ 3 & 2 & 0 & 1 & 16 \\ 2 & 0 & 12 & -5 & 5 \end{array}\right]$$

As noted in the algorithm, we work column by column. What is the first column that contains the first non-zero entry of a row? Clearly, column 1 contains the first non-zero entry of rows $1, 2$ and 3. Therefore, we need to get the first column into shape. We need to get a pivotal 1 into this column. Therefore, we do:

$$\left[\begin{array}{cccc|c} 2 & 1 & 3 & 0 & 16 \\ 3 & 2 & 0 & 1 & 16 \\ 2 & 0 & 12 & -5 & 5 \end{array}\right] \xrightarrow{R_2:R_2-R_1} \left[\begin{array}{cccc|c} 2 & 1 & 3 & 0 & 16 \\ 1 & 1 & -3 & 1 & 0 \\ 2 & 0 & 12 & -5 & 5 \end{array}\right]$$

We now have a pivotal 1 in row 2. Use this 1 to cancel the other non-zero entries in column 1:

$$\left[\begin{array}{cccc|c} 2 & 1 & 3 & 0 & 16 \\ 1 & 1 & -3 & 1 & 0 \\ 2 & 0 & 12 & -5 & 5 \end{array}\right] \xrightarrow{R_1:R_1-2R_2} \left[\begin{array}{cccc|c} 0 & -1 & 9 & -2 & 16 \\ 1 & 1 & -3 & 1 & 0 \\ 2 & 0 & 12 & -5 & 5 \end{array}\right]$$

$$\xrightarrow{R_3:R_3-2R_2} \left[\begin{array}{cccc|c} 0 & -1 & 9 & -2 & 16 \\ 1 & 1 & -3 & 1 & 0 \\ 0 & -2 & 18 & -7 & 5 \end{array}\right]$$

Finally, do a row swap to get the pivotal 1 in the correct position:

$$\left[\begin{array}{cccc|c} 0 & -1 & 9 & -2 & 16 \\ 1 & 1 & -3 & 1 & 0 \\ 0 & -2 & 18 & -7 & 5 \end{array}\right] \xrightarrow{\text{Swap } R_1,R_2} \left[\begin{array}{cccc|c} 1 & 1 & -3 & 1 & 0 \\ 0 & -1 & 9 & -2 & 16 \\ 0 & -2 & 18 & -7 & 5 \end{array}\right]$$

Now, on to the next column. The leftmost column that contains the first non-zero entry of a row is column 2, since it contains the first non-zero entries of both rows 2 and 3. To get a pivotal 1 in this column, it suffices to multiply row 2 by $-1$. Then we can continue to use Row 2 to cancel out other non-zero entries in column 1. Proceeding:

$$\left[\begin{array}{cccc|c} 1 & 1 & -3 & 1 & 0 \\ 0 & -1 & 9 & -2 & 16 \\ 0 & -2 & 18 & -7 & 5 \end{array}\right] \xrightarrow{R_2:R_2\times(-1)} \left[\begin{array}{cccc|c} 1 & 1 & -3 & 1 & 0 \\ 0 & 1 & -9 & 2 & -16 \\ 0 & -2 & 18 & -7 & 5 \end{array}\right]$$

$$\xrightarrow{R_1:R_1-R_2} \left[\begin{array}{cccc|c} 1 & 0 & 6 & -1 & 16 \\ 0 & 1 & -9 & 2 & -16 \\ 0 & -2 & 18 & -7 & 5 \end{array}\right]$$

$$\xrightarrow{R_3:R_3+2R_2} \left[\begin{array}{cccc|c} 1 & 0 & 6 & -1 & 16 \\ 0 & 1 & -9 & 2 & -16 \\ 0 & 0 & 0 & -3 & -27 \end{array}\right]$$

Now, moving on to the next column. Note that this column is **not** column 3: column 3 doesn't contain the first non-zero entry of any row! The only remaining column containing the first non-zero entry of a row is column 4, and this only contains the first non-zero entry of row 3. Therefore, this row will have to by definition contain our pivotal 1. Start by diving this row by $-3$, and proceed like above:

$$\left[\begin{array}{cccc|c} 1 & 0 & 6 & -1 & 16 \\ 0 & 1 & -9 & 2 & -16 \\ 0 & 0 & 0 & -3 & -27 \end{array}\right] \xrightarrow{R_3:R_3/(-3)} \left[\begin{array}{cccc|c} 1 & 0 & 6 & -1 & 16 \\ 0 & 1 & -9 & 2 & -16 \\ 0 & 0 & 0 & 1 & 9 \end{array}\right]$$

$$\xrightarrow{R_1:R_1+R_3} \left[\begin{array}{cccc|c} 1 & 0 & 6 & 0 & 25 \\ 0 & 1 & -9 & 2 & -16 \\ 0 & 0 & 0 & 1 & 9 \end{array}\right]$$

$$\xrightarrow{R_2:R_2-2R_3} \left[\begin{array}{cccc|c} 1 & 0 & 6 & 0 & 25 \\ 0 & 1 & -9 & 0 & -34 \\ 0 & 0 & 0 & 1 & 9 \end{array}\right]$$

The matrix is now in row-reduced echelon form. Rewriting this as a system of equations again, we get:

$$\begin{aligned} x_1 \qquad + 6x_3 \quad &= 25 \\ + x_2 - 9x_3 \quad &= -34 \\ x_4 &= 9 \end{aligned}$$

Recalling that the variables corresponding to pivotal columns are dependent, while the remaning variables are independent, we see that the only independent variable is $x_3$. Solving for everything in terms of $x_3$:

$$\begin{aligned} x_1 &= 25 - 6x_3 \\ x_2 &= -34 + 9x_3 \\ x_3 &= x_3 \\ x_4 &= 9 \end{aligned}$$

Therefore, all solutions can be written in the form above: writing this concisely as a vector, we see that solutions are all of the form $[25-6x_3, -34+9x_3, x_3, 9]$.

**Note:** If you want, check that this works by plugging it back into the original system, just like in the solution above!

3.

$$\begin{aligned} x_1 + 2x_2 + x_3 \;\; &= 0 \\ -x_1 \qquad + 2x_3 &= 1 \\ x_1 + 4x_2 + 4x_3 &= 2 \end{aligned}$$

**Solution:** Rewriting this in augmented matrix form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -1 & 0 & 2 & 1 \\ 1 & 4 & 4 & 2 \end{array}\right]$$

Now, using Row 1 to get column 1 in shape:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -1 & 0 & 2 & 1 \\ 1 & 4 & 4 & 2 \end{array}\right] \xrightarrow{R_2:R_2+R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 2 & 3 & 1 \\ 1 & 4 & 4 & 2 \end{array}\right]$$
$$\xrightarrow{R_3:R_3-R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 2 & 3 & 2 \end{array}\right]$$

The next column that contains the non-zero entry of a row is column 2. In order to get a pivotal 1 into this column, divide row 2 by 2:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 2 & 3 & 2 \end{array}\right] \xrightarrow{R_2:R_2/2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & 1/2 \\ 0 & 2 & 3 & 2 \end{array}\right]$$

(Note that if you pay attention, it's already clear at the last step that there are no solutions: why? We're still going to continue row-reducing, just to see what happens.)

Now, use Row 2 to cancel out the non-zero entries in column 2:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & 1/2 \\ 0 & 2 & 3 & 2 \end{array}\right] \xrightarrow{R_3:R_3-2R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & 1/2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$
$$\xrightarrow{R_1:R_1-2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 3/2 & 1/2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

This is now in row-reduced echelon form. The equations correspond to:

$$\begin{aligned} x_1 \quad + x_3 &= 0 \\ x_2 - 2x_3 &= -1 \\ 0 &= 1 \end{aligned}$$

Clearly, the last equation is impossible, implying that there are no solutions to the system of equations.