Solutions to Systems of Equations from September 20th

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Find all solutions to the following systems of equations:

1.

$$\begin{aligned} -5a - 2b + 2c &= 14 \\ 3a + b \;\; - c \;\; &= -8 \\ 2a + 2b - c \;\; &= -3 \end{aligned}$$

Solution: Putting this in augmented matrix form, we get

$$\left[\begin{array}{ccc|c} -5 & -2 & 2 & 14 \\ 3 & 1 & -1 & -8 \\ 2 & 2 & -1 & -3 \end{array}\right]$$

We work on the columns from left to right. The leftmost unfinished column that contains the first non-zero entry of a row is column 1, since it contains the first non-zero entry of every row. We need to get a pivotal 1 in this column. In order to avoid fractions, we start with:

$$\left[\begin{array}{ccc|c} -5 & -2 & 2 & 14 \\ 3 & 1 & -1 & -8 \\ 2 & 2 & -1 & -3 \end{array}\right] \xrightarrow{R_2 : R_2 - R_3} \left[\begin{array}{ccc|c} -5 & -2 & 2 & 14 \\ 1 & -1 & 0 & -5 \\ 2 & 2 & -1 & -3 \end{array}\right]$$

We now have a pivotal 1 in column 1 in row 2. Let's use this pivotal 1 to cancel out the non-zero entries of the first column (we could also start by swapping the pivotal 1 into the correct place – at the end, we will want the pivotal 1 in the first, not second, row.)

$$\left[\begin{array}{ccc|c} -5 & -2 & 2 & 14 \\ 1 & -1 & 0 & -5 \\ 2 & 2 & -1 & -3 \end{array}\right] \xrightarrow{R_1 : R_1 + 5R_2} \left[\begin{array}{ccc|c} 0 & -7 & 2 & -11 \\ 1 & -1 & 0 & -5 \\ 2 & 2 & -1 & -3 \end{array}\right]$$

$$\xrightarrow{R_3 : R_3 - 2R_3} \left[\begin{array}{ccc|c} 0 & -7 & 2 & -11 \\ 1 & -1 & 0 & -5 \\ 0 & 4 & -1 & 7 \end{array}\right]$$

Finally, swap the pivotal 1 into the correct place:

$$\left[\begin{array}{ccc|c} 0 & -7 & 2 & -11 \\ 1 & -1 & 0 & -5 \\ 0 & 4 & -1 & 7 \end{array}\right] \xrightarrow{\text{swap } R_1, R_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -5 \\ 0 & -7 & 2 & -11 \\ 0 & 4 & -1 & 7 \end{array}\right]$$

The first column is now done. The next column that contains the first non-zero entry of a row is column 2, which contains the first non-zero entries of both rows 2 and 3. To get a 1 in row 2 without fractions (and without using row 1 – that would mess up the first column!), we do the following:

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -5 \\ 0 & -7 & 2 & -11 \\ 0 & 4 & -1 & 7 \end{array}\right] \xrightarrow{R_2:R_2+2R_3} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -5 \\ 0 & 1 & 0 & 3 \\ 0 & 4 & -1 & 7 \end{array}\right]$$

Now use the pivotal 1 in row 2 to cancel out the other non-zero entries in column 2:

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -5 \\ 0 & 1 & 0 & 3 \\ 0 & 4 & -1 & 7 \end{array}\right] \xrightarrow{R_3:R_3-4R_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & -5 \end{array}\right]$$

$$\xrightarrow{R_1:R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & -5 \end{array}\right]$$

Finally, the only thing remaining is to get the last column in order. Clearly, to do so we just multiply row 3 by -1:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & -5 \end{array}\right] \xrightarrow{R_3:R_3\times(-1)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Therefore, at this point the system of equations corresponds to (recalling the names of the variables that we started with):

$$\begin{aligned} a &= -2 \\ b &= 3 \\ c &= 5 \end{aligned}$$

and so this is the only solution to the system of equations.

2.

$$\begin{aligned} 3x_1 - 2x_2 &= -54 - 4x_3 \\ -x_1 - 2x_3 &= 20 - x_2 \\ 5x_1 - 4x_2 + 8x_3 &= -83 \end{aligned}$$

Solution: Rewriting this in the familiar form with all the variables in order on one side and all the numbers on the other, we get

$$\begin{aligned}3x_1 - 2x_2 + 4x_3 &= -54\\ -x_1 + x_2 \quad - 2x_3 &= 20\\ 5x_1 - 4x_2 + 8x_3 &= -83\end{aligned}$$

In augmented matrix form, this is:

$$\left[\begin{array}{ccc|c} 3 & -2 & 4 & -54\\ -1 & 1 & -2 & 20\\ 5 & -4 & 8 & -83\end{array}\right]$$

Now, work on column 1. Start by getting a pivotal 1 in row 2, and proceed by using it to cancel non-zero elements of column 1.

$$\left[\begin{array}{ccc|c} 3 & -2 & 4 & -54\\ -1 & 1 & -2 & 20\\ 5 & -4 & 8 & -83\end{array}\right] \xrightarrow{R_2:R_2\times(-1)} \left[\begin{array}{ccc|c} 3 & -2 & 4 & -54\\ 1 & -1 & 2 & -20\\ 5 & -4 & 8 & -83\end{array}\right]$$

$$\xrightarrow{R_1:R_1-3R_2} \left[\begin{array}{ccc|c} 0 & 1 & -2 & 6\\ 1 & -1 & 2 & -20\\ 5 & -4 & 8 & -83\end{array}\right]$$

$$\xrightarrow{R_3:R_3-5R_2} \left[\begin{array}{ccc|c} 0 & 1 & -2 & 6\\ 1 & -1 & 2 & -20\\ 0 & 1 & -2 & 17\end{array}\right]$$

$$\xrightarrow{\text{swap } R_1,\ R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & -20\\ 0 & 1 & -2 & 6\\ 0 & 1 & -2 & 17\end{array}\right]$$

Column 1 is now done! (It is now also clear that this has no solutions, since the last two rows correspond to equations that are clearly impossible at the same time. Nonetheless, we shall continue row-reducing to see what happens.) Now, work on column 2 – use the pivotal 1 in row 2:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -20\\ 0 & 1 & -2 & 6\\ 0 & 1 & -2 & 17\end{array}\right] \xrightarrow{R_1:R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -14\\ 0 & 1 & -2 & 6\\ 0 & 1 & -2 & 17\end{array}\right]$$

$$\xrightarrow{R_3:R_3-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -14\\ 0 & 1 & -2 & 6\\ 0 & 0 & 0 & 11\end{array}\right]$$

This is now in row-reduced echelon form. The last row corresponds to the equation $0 = 11$, and therefore the system has no solutions.

3.

$$c_1\begin{bmatrix}-2\\7\\3\end{bmatrix} + c_2\begin{bmatrix}1\\-2\\-1\end{bmatrix} + c_3\begin{bmatrix}8\\-22\\-10\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Solution: Simplifying the right-hand side using the rules of scalar multiplication and addition results in:

$$\begin{bmatrix} -2c_1 \\ 7c_2 \\ 3c_1 \end{bmatrix} + \begin{bmatrix} c_2 \\ -2c_2 \\ -c_2 \end{bmatrix} \begin{bmatrix} 8c_3 \\ -22c_3 \\ -10c_3 \end{bmatrix} = \begin{bmatrix} -2c_1 + c_2 + 8c_3 \\ 7c_1 - 2c_2 - 22c_3 \\ 3c_2 - c_2 - 10c_3 \end{bmatrix}$$

Therefore, the system of equations works out to be:

$$\begin{bmatrix} -2c_1 + c_2 + 8c_3 \\ 7c_1 - 2c_2 - 22c_3 \\ 3c_2 - c_2 - 10c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which, coordinate by coordinate, is precisely

$$\begin{aligned} -2c_1 + c_2 \quad + 8c_3 \quad &= 0 \\ 7c_1 - 2c_2 - 22c_3 &= 0 \\ 3c_1 - c_2 \quad - 10c_3 &= 0 \end{aligned}$$

Finally, writing this in augmented matrix form:

$$\left[\begin{array}{ccc|c} -2 & 1 & 8 & 0 \\ 7 & -2 & -22 & 0 \\ 3 & -1 & -10 & 0 \end{array}\right]$$

We get a 1 in the first column in the second row, then use to cancel out the non-zero entries in the first column:

$$\left[\begin{array}{ccc|c} -2 & 1 & 8 & 0 \\ 7 & -2 & -22 & 0 \\ 3 & -1 & -10 & 0 \end{array}\right] \xrightarrow{R_2:R_2-2R_3} \left[\begin{array}{ccc|c} -2 & 1 & 8 & 0 \\ 1 & 0 & -2 & 0 \\ 3 & -1 & -10 & 0 \end{array}\right]$$

$$\xrightarrow{R_1:R_1+2R_2} \left[\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 1 & 0 & -2 & 0 \\ 3 & -1 & -10 & 0 \end{array}\right]$$

$$\xrightarrow{R_3:R_3-3R_2} \left[\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & -1 & -4 & 0 \end{array}\right]$$

$$\xrightarrow{\text{Swap } R_1,\ R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -1 & -4 & 0 \end{array}\right]$$

Now, to work on the second column, using the pivotal 1 already in the second row:

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -1 & -4 & 0 \end{array}\right] \xrightarrow{R_2:R_2/7} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -1 & -4 & 0 \end{array}\right]$$

$$\xrightarrow{R_2:R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is now in row-reduced echelon form. This corresponds to the following system of equations:

$$\begin{aligned} c_1 \quad - 2c_3 &= 0 \\ c_2 + 4c_3 &= 0 \\ 0 \quad &= 0 \end{aligned}$$

The columns containing the pivots are the first and second column, and the column without a pivot is the third column. Therefore, c_1 and c_2 are dependent variables, whereas c_3 is the independent variable. Thus, we solve for everything in terms of c_3, getting:

$$\begin{aligned} c_1 &= 2c_3 \\ c_2 &= -4c_3 \\ c_3 &= c_3 \end{aligned}$$

Therefore, all solutions to this system of equations are $[c_1, c_2, c_3] = [2c_3, -4c_3, c_3]$.