Final: Concepts to Review

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The final covers everything we did in class so far – it is cumulative. However, there will be more emphasis on the things covered since the last midterm: that is, Sections 4.1, 4.3, 4.4, 4.7, 4.9, 5.1, 5.2, and 5.3.

The review below is composed of the earlier review sheets, with the added material from after the second midterm.

1. Trigonometry (Appendix D)
   - Using the unit circle to find $\sin(\theta)$ and $\cos(\theta)$
   - Definitions of the trig functions, such as $\sin, \cos, \tan, \cot, \csc, \sec$
   - Formulas such as

   $$\begin{aligned} \sin^2(\theta) + \cos^2(\theta) &= 1 \\ \tan^2(\theta) + 1 &= \sec^2(\theta) \\ 1 + \cot^2(\theta) &= \csc^2(\theta) \end{aligned}$$

   - Angle addition formulas like $\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$, and versions for $\cos, \tan$, etc.
   - Values of the trig functions at the angles in the table on page A27: that is, multiples of $\frac{\pi}{4}$ and multiples of $\frac{\pi}{6}$
2. Exponentials (Section 1.5)
   - The definition of $f(x) = a^x$.
   - Laws of exponents (and using them to solve problems)

   $$a^{x+y} = a^x a^y, \; a^{x-y} = \frac{a^x}{a^y}, \; a^{xy} = (a^x)^y, \; (ab)^x = a^x b^x$$

   - Graphs of exponential functions
3. Inverse Functions and Logs (Section 1.6)
   - One-to-one functions
   - The definition of $f^{-1}(x)$ if $f(x)$ is one-to-one, domain and range of f^{-1}, graphing f^{-1} given the graph of f

- Cancellation equations:

$$f(f^{-1}(x)) = x,\ f^{-1}(f(x)) = x$$

- Finding an explicit formula for the inverse of a function
- Logarithms, logarithm laws, the number e and the natural log (ln):

$$\log_a(xy) = \log_a(x) + \log_a(y), \quad \log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y),$$
$$\log_a(x^r) = r\log_a(x), \quad \log_a(x) = \frac{\ln x}{\ln a}$$

- Inverse trigonometric functions: definitions and graphs

4. The limit of a function (Section 2.2)

- The concept of a limit: what it means when

$$\lim_{x\to a} f(x) = L$$

- Using the graph of a function to determine a limit
- One-sided limits (knowing what the following mean):

$$\lim_{x\to a^+} f(x) = L, \lim_{x\to a^-} f(x) = L$$

- If $f(x) \le g(x)$ near (except possibly at) a, and both the limits exist, then

$$\lim_{x\to ax} f(x) \le \lim_{x\to a} g(x)$$

- The Squeeze Theorem: if $f(x) \le g(x) \le h(x)$ near (except possibly at) a, and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then

$$\lim_{x\to a} g(x) = L$$

- Infinite limits: the meanings of

$$\lim_{x\to a} f(x) = \infty, \lim_{x\to a} f(x) = -\infty$$

and variants with one-sided limits, such as $\lim_{x\to a^+} f(x) = \infty$.
- Vertical asymptotes of functions

5. Calculating limits using limit laws (Section 2.3)

- If c is a constant, and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then:

$$\begin{aligned}
\lim_{x\to a}(f(x)+g(x)) &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x)\\
\lim_{x\to a}(f(x)-g(x)) &= \lim_{x\to a} f(x) - \lim_{x\to a} g(x)\\
\lim_{x\to a}(cf(x)) &= c\lim_{x\to a} f(x)\\
\lim_{x\to a}(f(x)g(x)) &= \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)\\
\lim_{x\to a}\frac{f(x)}{g(x)} &= \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \text{ if } \lim_{x\to a} g(x)\neq 0
\end{aligned}$$

- Other laws that follow from above in Section 2.3 (see textbook for full list)
- Being able to use the above laws to calculate limits
- Knowing when *not* to use these limit laws: if $\lim_{x\to a} f(x)$ or $\lim_{x\to a} g(x)$ don't exist, we can't use these. Remember that just because these don't exist does NOT mean that, say, $\lim_{x\to a}(f(x)+g(x))$ doesn't exist– you just can't use the laws to calculate it!

6. Continuity (Section 2.5)

- The definition of being continuous at a point: $f(x)$ is continuous at a if
$$\lim_{x\to a} f(x) = f(a)$$
- Similar definitions for $f(x)$ being continuous at a from the left and from the right.
- Using the graph of a function to determine where it's continuous
- If f and g are continuous at a, then the following functions are also continuous at a:
$$f+g,\ f-g,\ fg,\ \frac{f}{g} \text{ if } g(a)\neq 0$$
- Knowing functions that are continuous everywhere on their domain, and using this fact to calculate limits. These include: polynomials, rational functions, root functions, exponentials, logarithms, trig functions, inverse trig functions.
- If f is continuous at b, and $\lim_{x\to a} g(x) = b$, then
$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b)$$
- Using the Intermediate Value Theorem to show that equations have a solution (without finding the solution!)

7. Limit at infinity (Section 2.6)

- The meanings of

$$\lim_{x\to\infty} f(x) = L, \quad \lim_{x\to-\infty} f(x) = L$$

- Horizontal asymptotes of functions
- Rules such as

$$\lim_{x\to\infty} \frac{1}{x^r} = 0$$

  for rational numbers $r > 0$.
- Manipulating expressions in order to calculate limits at infinity

8. Calculating derivatives using limits (Sections 2.7)

   - $f'(a)$ is defined to be the slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$.
   - $f'(a)$ is also the instantaneous rate of change of $f(x)$ at $x = a$.
   - The limit definition of the derivative is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

   - Finding the equation of a tangent line to $y = f(x)$ at $(a, f(a))$ using the derivative.

9. The derivative as a function (Section 2.8)

   - Just like above, the definition of $f'(x)$ is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

     This is a function of x.
   - What it means for a function to be differentiable at a and on an interval
   - How a function can fail to be differentiable (some possibilities: a corner, a disconuity, or a vertical tangent)
   - Higher derivatives: $f''(x)$ is the derivative of $f'(x)$, $f^{(n)}(x)$ is the nth derivative of x which is defined to be the derivative of $f^{(n-1)}(x)$.
   - Graphing $f'(x)$ and $f''(x)$ given a graph of $f(x)$

10. Differentiation Rules (Section 3.1)

    - Derivatives of constant functions and powers of x:

$$(c)' = 0$$
$$(x^n) = nx^{n-1}$$

- The sum, difference, and constant multiple rules:

$$(f(x)+g(x))' = f'(x)+g'(x)$$
$$(f(x)-g(x))' = f'(x)-g'(x)$$
$$(cf(x))' = cf'(x)$$

- The derivative of e^x:

$$(e^x)' = e^x$$

11. The Product and Quotient Rules (Section 3.2)

- The product rule:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

- The quotient rule:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Be careful with the order of the terms in the numerator!!

12. Derivatives of trig functions (Section 3.3:)

- The main formulas:

$$(\sin(x))' = \cos(x)$$
$$(\cos(x))' = -\sin(x)$$

- The following derivatives can either be memorized or figured out using differentiation rules:

$$(\tan(x))' = \sec^2(x)$$
$$(\cot(x))' = -\csc^2(x)$$
$$(\csc(x))' = -\csc(x)\cot(x)$$
$$(\sec(x))' = \sec(x)\tan(x)$$

13. The Chain Rule (Section 3.4)

- If $F(x) = f(g(x))$, then

$$F'(x) = f'(g(x)) \cdot g'(x)$$

- Alternatively with boxes, if $F(x) = f(\Box)$, then

$$F'(x) = f'(\Box) \cdot (\text{Derivative of what's inside } \Box)$$

- The following rule follows from the chain rule:

$$(a^x)' = \ln(a)a^x$$

14. Implicit Differentiation (Section 3.5)

    - Using the chain rule to find y' given a relationship between x and y: e.g., find $y' = \frac{dy}{dx}$ in terms of x and y if $x^2 + y^2 = xy$.
    - Substituting in the original relationship between x and y in order to simplify y'.
    - Derivatives of inverse trig functions:

$$\begin{aligned}
(\arcsin(x))' &= \frac{1}{\sqrt{1-x^2}} \\
(\arccos(x))' &= -\frac{1}{\sqrt{1-x^2}} \\
(\arctan(x))' &= \frac{1}{1+x^2} \\
(\operatorname{arccot}(x))' &= -\frac{1}{1+x^2} \\
(\operatorname{arccsc}(x))' &= -\frac{1}{x\sqrt{x^2-1}} \\
(\operatorname{arcsec}(x))' &= \frac{1}{x\sqrt{x^2-1}}
\end{aligned}$$

15. Derivatives of logarithmic functions and logarithmic differentiation (Section 3.6)

    - The rule for differentiating $\ln(x)$:

$$(\ln(x))' = \frac{1}{x}$$

    - Differentiating a log with another base:

$$(\log_a(x))' = \frac{1}{x\ln(a)}$$

    - Logarithmic differentiation: if $y = f(x)$ is written with a lot of products, quotients, and exponents, you can do the following:
      (a) Take the ln of both sides and simplify using log rules.
      (b) Differentiate implicitly with respect to x.
      (c) Solve for y', then substitute the original expression for y to get the answers in terms of x.
    - An example where logarithmic differentiation would be useful: differentiate $y = \sin(x)^{\cos(x)} \cdot e^x$.

- Make sure to use the log rules correctly! You can get all sorts of wrong answers by using 'identities' like $\ln(x+y) = \ln(x) + \ln(y), \ln(x)^r = r\ln(x)$, etc.

16. Related rates (Section 3.9)
    - In related rates, all functions are in terms of time! When we write y' here, what we mean is $\frac{dy}{dt}$.
    - Our algorithm for related rates from class:
        (a) Draw the picture **at an arbitrary time**.
        (b) Give names to all the relevant variables. Note that this will require making choices. Keep in mind the next step – it should be easy to write down what you're given and what you're looking for in terms of your choices!
        (c) Write down what you're given, and what you're looking for.
        (d) Find all the relationships between your variables.
        (e) Differentiate the relationship(s) using implicit differentiation.
        (f) Plug in the instantaneous information given (making sure to solve for all the relevant quantities at that instant) to find what we need.
    - Common related rates problems:
        (a) Two ships (cars, people, etc.) moving away from each other, their speed given – how quickly is the distance changing?
        (b) Ladder sliding down a wall.
        (c) Shadow problems (person walking away from streetlight, etc.)
        (d) Volume and surface area growth problems.
        (e) Point moving along a specified graph problems.

17. Linearizations (or linear approximations) (Section 3.10)
    - The linearization $L(x)$ of the function $y = f(x)$ at $x = a$ is defined to be
    $$L(x) = f(a) + f'(a)(x - a)$$
    - $y = L(x)$ is the equation of the tangent line to $y = f(x)$ at $(a, f(a))$ (you could use this fact to calculate $L(x)$ if you've forgotten the formula!)
    - For values of x that are close to a, $L(x)$ is close to $f(x)$; this allows us to use $L(x)$ to estimate $f(x)$. For example, we could estimate $\sqrt{4.1}$ using the linearization of $f(x) = \sqrt{x}$ at $x = 4$.

18. Maximum and Minimum Values (Section 4.1):
    - Definition of absolute minimum and maximum
    - Definition of local minimum and maximum

- c is a critical point (or critical number) of f if it's in the domain of f, and $f'(c)$ is either 0 or doesn't exist.
- The Closed Interval Test for finding the absolute extrema of a continuous function f on $[a, b]$:
  (a) Find all the critical points of f.
  (b) Plug in the critical points of f as well as the endpoints a and b into f. The smallest value you get is the absolute minimum of f on $[a, b]$; the largest value you get the absolute maximum of f on $[a, b]$.

19. How Derivatives Affect the Shape of a Graph (Section 4.3)

- Using f' to determine whether f is increasing or decreasing: $f'(x) > 0$ means that f is increasing, $f'(x) < 0$ means that f is decreasing.
- The First Derivative Test: If c is a critical point of a continuous function f, then
  (a) If f' changes from positive to negative at c, then f has a local max at c.
  (b) If f' changes from negative to positive at c, then f has a local min at c.
  (c) If f' doesn't change sign, then there's neither a local min or a local max.
- Definition (and picture) for functions that are concave up and down
- Using f'' to determine concavity: $f''(x) > 0$ means concave up, $f''(x) < 0$ means concave down.
- The Second Derivative Test: if f'' is continuous near c, then
  (a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local min at c.
  (b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local max at c.
- Note that the Second Derivative Test doesn't apply at critical points where $f'(c)$ DNE. Furthermore, if $f'(c)$ exists but $f''(c) = 0$, we have no information.
- Using f', f'', and asymptotes (which you should know how to do from earlier!) to sketch graphs of functions.

20. Indeterminate Forms and L'Hospital's Rule (Section 4.4)

- If we have a limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ (where the ∞ can stand for either $-\infty$ or $+\infty$), then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

- Before using L'Hospital's, ALWAYS make sure first that it's in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, otherwise you may well use it to get a wrong answer!

- If we have a limit of a product $\lim_{x \to a} f(x)g(x)$ which is of the form $0 \cdot \infty$, we can rearrange it as

$$\lim_{x \to a} \frac{f(x)}{1/g(x)} \text{ or } \lim_{x \to a} \frac{g(x)}{1/f(x)}$$

and then use L'Hospital's the way we always do for a quotient.

- If we have a limit of a power $\lim_{x \to a} f(x)^{g(x)}$, which is of the form 0^0, or ∞^0, or 1^∞, then we can do the following steps:
  (a) Let $L = \lim_{x \to a} f(x)^{g(x)}$. Then, taking ln of both sides we get that $\ln(L) = \lim_{x \to a} g(x) \ln(f(x))$.
  (b) Use the standard L'Hospital's Rule tricks to calculate

$$\lim_{x \to a} g(x) \ln(f(x))$$

  (c) The answer you calculated in the last step is $\ln(L)$; therefore, to get the actual answer L, take e to the power of the answer in (b).

- Sometimes you will have to use L'Hospital's more than once to get the answer. Also, beware of questions which don't require L'Hospital's and don't forget the other limit rules!

21. Optimization Problems (Section 4.7):

- These are word problems where you're asked to show the maximal or minimal value of something explained in the problem. Here are the steps for these problems:
  (a) Draw a diagram if necessary.
  (b) Label all the relevant variables. Make sure to give a name to the quantity being maximized or minimized!
  (c) Express the quantity maximized or minimized in terms of the other variables.
  (d) Find relationships between the other variables. Use these relationship and substitution to express the quantity being maximized or minimized in terms of just ONE variable.
  (e) Find the domain of this function, given the variable it's defined in terms of. If the variables have a geometric meaning, you should think about what domain makes sense!
  (f) Use tests for absolute extrema to find the absolute minimum or maximum of the function.
- When the domain of the function is a closed interval, you can use the Closed Interval Test.
- The First Derivative Test for Absolute Extrema can also be very useful: if c is a critical point of a continuous function f defined on an interval, then

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.
(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

22. Antiderivatives (Section 4.9):

- The definition of an antiderivative: $F(x)$ is an antiderivative of $f(x)$ on an interval I if $F'(x) = f(x)$ for all x in I.
- Some common antiderivatives – there's a table on page 345.
- Finding the general form of an antiderivative
- Finding an antiderivative satisfying certain conditions (solving for the constant in the antiderivative using the conditions.)
- Finding a function $F(x)$ such that a $F''(x) = f(x)$ under certain conditions – that is, taking an antiderivative twice (or even more times.)

23. Areas and Integrals (Sections 5.1 and 5.2):

- Using right endpoints and left endpoints to estimate an area under a curve
- The definition of $\int_a^b f(t)\,dt$ as the "net area" under the curve $y = f(t)$ from $x = a$ to $x = b$.
- Using a Riemann sum with arbitrary sample points to estimate an integral. In particular, using the midpoint rule to estimate an integral.
- Using sigma notation to express summations
- Taking a limit of Riemann sums to calculate an integral.
- Properties of the definite integral:

(a) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
(b) $\int_a^a f(x)\,dx = 0$
(c) $\int_a^b c\,dx = c(b-a)$
(d) $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
(e) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
(f) $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$
(g) $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int a^b f(x)\,dx$

24. The Fundamental Theorem of Calculus (Section 5.3):

- The FTC, part 1: If f is continuous on $[a,b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \quad a \le x \le b$$

is continuous on $[a, b]$ and differentiable on (a, b), and

$$g'(x) = f(x)$$

- The FTC, part 2: If f is continuous on $[a, b]$, and $F(x)$ is any antiderivative of f, then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$