MATH 408N PRACTICE FINAL

Name:____________________

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TA session: ____________________

Show your work for all the problems. Good luck!

(1) Let $f(x) = \frac{e^x}{e^x - 1}$.

(a) [5 pts] State the domain and range of $f(x)$.

(b) [5 pts] Calculate a formula for $f^{-1}(x)$.

(c) [5 pts] Find the domain and range of f^{-1}.

(2) Calculate the following limits, using whatever tools are appropriate. State which results you're using for each question.

(a) [5 pts] $\lim_{x\to 1} \dfrac{x^2+1}{x-2}$

(b) [5 pts] $\lim_{x\to 0} \dfrac{\frac{1}{x} - \frac{1}{x-1}}{x^{-1}}$

(c) [5 pts] $\lim_{x\to 0} \dfrac{\cos(x)-1}{e^x - x - 1}$

(d) [5 pts] $\lim_{x \to 2} f(x)$, where $2 \leq f(x) \leq x^2 - 2$ for all $x \in [1, 4]$.

(e) [5 pts] $\lim_{x \to \infty} \left(1 - \frac{1}{x}\right)^{3x+1}$

(f) $\lim_{x \to 4} \frac{x - 4}{\sqrt{x} - 2}$

(3) Let the function $f(x)$ be defined piecewise as follows:

$$f(x) = \begin{cases} -1 & x \leq -1 \\ x^2 & -1 < x < 1 \\ x & x \geq 1 \end{cases}$$

(a) [5 pts] Sketch a graph of this function.

(b) [10 pts] State the intervals on which $f(x)$ is continuous. Do a limit calculation checking for continuity at any points where this is necessary.

(4) Calculate the following derivatives using the limit definition of the derivative. You may NOT use L'Hospital's rule for these.

(a) [5 pts] $f'(x)$, where $f(x) = x^2 - 2$.

(b) [5 pts] $f'(1)$, where $f(x) = \frac{1}{\sqrt{x}}$.

(5) Calculate the following derivatives using whichever tools you wish. State the results you're using. You do NOT need to simplify your answers!

(a) [5 pts] Find $f'(x)$, if $f(x) = \ln(x)e^x + \sin(x)$.

(b) [5 pts] Find $f'(x)$, if $f(x) = \dfrac{\tan(e^x)}{x^2+1}$

(c) [5 pts] Find $f'(x)$, if $f(x) = x^2\cos(x)^{\sin(x)+1}$

(d) [5 pts] Find y' in terms of x and y, if $xy + e^y = \arctan(x)$.

(e) [5 pts] Find $g'(x)$, if $g(x) = \arccos(x) \cdot \int_1^x e^{t^2} \sin(\cos(t))\, dt$

(f) [5 pts] Find $g'(x)$, if $g(x) = \int_1^{x^2+1} (u^2 + u)\, du$

(6) Calculate the equations of the following tangent lines:

(a) The tangent line to $y = \dfrac{e^{x-1}}{\ln(x)+1}$ at $x = 1$.

(b) The tangent line to $y = f(x)g(x)$ at $x = 0$, given that $f(0) = 2, g(0) = 3, f'(0) = -1$, and $g'(0) = 4$.

(7) (a) [5 pts] Find the linearization of $f(x) = x^{1/3}$ at $x = 27$.

(b) [5 pts] Use the result from part (a) to estimate $\sqrt[3]{29}$.

(c) [5 pts] Could you use the result from (a) to estimate $\sqrt[3]{65}$, or would you need to do something different? (If you need to do something different, please eplain what it is.)

(8) [10 pts] A sphere is expanding, with its volume growing at a rate of $4\text{ft}^3/\text{sec}$. How quickly is its surface area changing, when the volume of the sphere is 36π ft^3?

You may use the following formulas for the surface area and volume of a sphere with radius r:

$$A = 4\pi r^2, V = \frac{4}{3}\pi r^3$$

(9) [10 pts] Let $f(x) = x^3 + 6x^2 + 9x + 7$. Find the absolute minimum value and absolute maximum value of f on the interval $[-4, 2]$.

(10) Let $f(x) = \dfrac{e^x}{x-1}$. Answer the following questions about $f(x)$.

(a) [5 pts] Find all the critical points of $f(x)$.

(b) [5 pts] Find the intervals on which $f(x)$ is increasing and decreasing.

(c) [5 pts] Find the intervals on which $f(x)$ is concave up and concave down.

(d) [5 pts] Find the horizontal asymptotes of $f(x)$. For each asymptote, state whether it occurs at ∞ or $-\infty$.

(e) [5 pts] Find the vertical asymptotes of $f(x)$. For each vertical asymptotes $x = a$, calculate $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$.

(f) [5 pts] Use the information from the previous parts of the question to sketch the graph of $f(x)$.

(11) [10 pts] Find the point on the parabola $y = x^2 - 2$ that is closest to the origin (that is, to the point $(0, 0)$).

(12) Solve the following problems:

(a) [5 pts] Find the general expression for a function $F(x)$ such that $F'(x) = e^{2x} - \sin(x) + \frac{1}{1+x^2}$.

(b) [5 pts] Find the function $F(x)$ such that $F'(x) = 2x + 1$ and $F(1) = 3$.

(c) Find the function $F(x)$ such that $F''(x) = 1 + \frac{1}{x^2}$, with $F'(1) = 1$ and $F(1) = 2$.

(13) Solve the following problems:

(a) [5 pts] Estimate the area under $y = x^2$ from $x = 1$ to $x = 3$ using 4 rectangles and the right endpoint rule. Use the graph to explain whether this an underestimate or an overestimate.

(b) [5 pts] Estimate the area under $y = x^2$ from $x = 1$ to $x = 3$ using 4 rectangles and the left endpoint rule. Use the graph to explain whether this an underestimate or an overestimate.

(c) [5 pts] Estimate the area under $y = x^2$ from $x = 1$ to $x = 3$ using 4 rectangles and the midpoint rule. Is it immediately clear whether this is an underestimate or an overestimate?

(14) Solve the following problems:

(a) [5 pts] Express the sum $\frac{1}{4} + \frac{1}{5} + \cdot + \frac{1}{10}$ using sigma notation.

(b) [10 pts] Use the limit of Riemann sums with right endpoints to calculate the integral $\int_0^2 (x^2 + 1)\,dx$. You may use the formula

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

(15) Find the values of the following definite integrals, using whichever tools you choose. State the results you're using.

(a) [5 pts]

$$\int_{-1}^{2} e^x - x\,dx$$

(b) [5 pts]

$$\int_{\pi/6}^{\pi} \cos(x)\,dx$$

(c) [5 pts]

$$\int_{1}^{e} 1 + \frac{1}{x}\,dx$$