Midterm 2: Concepts to Review

Olena Bormashenko

The second midterm covers Section 2.7, 2.8, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.9, and 3.10 – everything we did in lecture starting on Tuesday, September 20th until Thursday, October 20th. Material from the first exam will not appear *explicitly* on the exams, but since we've been building on that material, you should know it! (For example, while we will not have straight limit questions, we will compute derivatives using limits; we won't ask you a trigonometry question directly, but it may very well come up in a related rates question.)

1. Calculating derivatives using limits (Sections 2.7)
   - $f'(a)$ is defined to be the slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$.
   - $f'(a)$ is also the instantaneous rate of change of $f(x)$ at $x = a$.
   - The limit definition of the derivative is

     $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

   - Finding the equation of a tangent line to $y = f(x)$ at $(a, f(a))$ using the derivative.
2. The derivative as a function (Section 2.8)
   - Just like above, the definition of $f'(x)$ is

     $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

     This is a function of x.
   - What it means for a function to be differentiable at a and on an interval
   - How a function can fail to be differentiable (some possibilities: a corner, a disconuity, or a vertical tangent)
   - Higher derivatives: $f''(x)$ is the derivative of $f'(x)$, $f^{(n)}(x)$ is the nth derivative of x which is defined to be the derivative of $f^{(n-1)}(x)$.
   - Graphing $f'(x)$ and $f''(x)$ given a graph of $f(x)$
3. Differentiation Rules (Section 3.1)

- Derivatives of constant functions and powers of x:

$$(c)' = 0$$
$$(x^n) = nx^{n-1}$$

- The sum, difference, and constant multiple rules:

$$(f(x) + g(x))' = f'(x) + g'(x)$$
$$(f(x) - g(x))' = f'(x) - g'(x)$$
$$(cf(x))' = cf'(x)$$

- The derivative of e^x:

$$(e^x)' = e^x$$

4. The Product and Quotient Rules (Section 3.2)

- The product rule:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

- The quotient rule:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Be careful with the order of the terms in the numerator!!

5. Derivatives of trig functions (Section 3.3:)

- The main formulas:

$$(\sin(x))' = \cos(x)$$
$$(\cos(x))' = -\sin(x)$$

- The following derivatives can either be memorized or figured out using differentiation rules:

$$(\tan(x))' = \sec^2(x)$$
$$(\cot(x))' = -\csc^2(x)$$
$$(\csc(x))' = -\csc(x)\cot(x)$$
$$(\sec(x))' = \sec(x)\tan(x)$$

6. The Chain Rule (Section 3.4)

- If $F(x) = f(g(x))$, then

$$F'(x) = f'(g(x)) \cdot g'(x)$$

- Alternatively with boxes, if $F(x) = f(\square)$, then

$$F'(x) = f'(\square) \cdot (\text{Derivative of what's inside } \square)$$

- The following rule follows from the chain rule:

$$(a^x)' = \ln(a)a^x$$

7. Implicit Differentiation (Section 3.5)
   - Using the chain rule to find y' given a relationship between x and y: e.g., find $y' = \frac{dy}{dx}$ in terms of x and y if $x^2 + y^2 = xy$.
   - Substituting in the original relationship between x and y in order to simplify y'.
   - Derivatives of inverse trig functions, and knowing how to derive them using implicit differentiation:

$$\begin{aligned}
(\arcsin(x))' &= \frac{1}{\sqrt{1-x^2}} \\
(\arccos(x))' &= -\frac{1}{\sqrt{1-x^2}} \\
(\arctan(x))' &= \frac{1}{1+x^2} \\
(\operatorname{arccot}(x))' &= -\frac{1}{1+x^2} \\
(\operatorname{arccsc}(x))' &= -\frac{1}{x\sqrt{x^2-1}} \\
(\operatorname{arcsec}(x))' &= \frac{1}{x\sqrt{x^2-1}}
\end{aligned}$$

8. Derivatives of logarithmic functions and logarithmic differentiation (Section 3.6)
   - The rule for differentiating $\ln(x)$:

$$(\ln(x))' = \frac{1}{x}$$

   - Differentiating a log with another base:

$$(\log_a(x))' = \frac{1}{x\ln(a)}$$

   - Logarithmic differentiation: if $y = f(x)$ is written with a lot of products, quotients, and exponents, you can do the following:
     (a) Take the ln of both sides and simplify using log rules.
     (b) Differentiate implicitly with respect to x.

(c) Solve for y', then substitute the original expression for y to get the answers in terms of x.

- An example where logarithmic differentiation would be useful: differentiate $y = \sin(x)^{\cos(x)} \cdot e^x$.
- Make sure to use the log rules correctly! You can get all sorts of wrong answers by using 'identities' like $\ln(x+y) = \ln(x) + \ln(y), \ln(x)^r = r\ln(x)$, etc.

9. Related rates (Section 3.9)

- In related rates, all functions are in terms of time! When we write y' here, what we mean is $\frac{dy}{dt}$.
- Our algorithm for related rates from class:
  (a) Draw the picture **at an arbitrary time**.
  (b) Give names to all the relevant variables. Note that this will require making choices. Keep in mind the next step – it should be easy to write down what you're given and what you're looking for in terms of your choices!
  (c) Write down what you're given, and what you're looking for.
  (d) Find all the relationships between your variables.
  (e) Differentiate the relationship(s) using implicit differentiation.
  (f) Plug in the instantaneous information given (making sure to solve for all the relevant quantities at that instant) to find what we need.
- Common related rates problems:
  (a) Two ships (cars, people, etc.) moving away from each other, their speed given – how quickly is the distance changing?
  (b) Ladder sliding down a wall.
  (c) Shadow problems (person walking away from streetlight, etc.)
  (d) Volume and surface area growth problems.
  (e) Point moving along a specified graph problems.

10. Linearizations (or linear approximations) (Section 3.10)

- The linearization $L(x)$ of the function $y = f(x)$ at $x = a$ is defined to be
$$L(x) = f(a) + f'(a)(x - a)$$
- $y = L(x)$ is the equation of the tangent line to $y = f(x)$ at $(a, f(a))$ (you could use this fact to calculate $L(x)$ if you've forgotten the formula!)
- For values of x that are close to a, $L(x)$ is close to $f(x)$; this allows us to use $L(x)$ to estimate $f(x)$. For example, we could estimate $\sqrt{4.1}$ using the linearization of $f(x) = \sqrt{x}$ at $x = 4$.