MATH 408N MIDTERM 1

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9/22/2011
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TA session: ____________________

Show your work for all the problems. Good luck!

(1) Let θ be an angle such that $\tan(\theta) = \frac{1}{2}$ and $\cos(\theta) < 0$.

(a) [5 pts] Calculate $\sec(\theta)$. (Please put this in the simplest form you can.)

Solution:

Recall that $1 + \tan^2(\theta) = \sec^2(\theta)$. Therefore,

$$\sec^2(\theta) = 1 + \left(\frac{1}{2}\right)^2 = 1 + \frac{1}{4} = \frac{5}{4}$$

Thus,

$$\sec(\theta) = \pm\sqrt{\frac{5}{4}} = \pm\frac{\sqrt{5}}{2}$$

Now we just need to figure out whether $\sec(\theta)$ is $\sqrt{5}/2$ or $-\sqrt{5}/2$. Since $\cos(\theta) < 0$, and $\sec(\theta) = 1/\cos(\theta)$, it must be true that $\sec(\theta) < 0$. Therefore,

$$\boxed{\sec(\theta) = -\frac{\sqrt{5}}{2}}$$

(b) [5 pts] Calculate $\sin(\theta)$. (This should also be put in simplest form.)

Solution:

This one can be done using part (a). Since

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \sin(\theta) \cdot \sec(\theta)$$

by moving things to the other side, we get

$$\sin(\theta) = \frac{\tan(\theta)}{\sec(\theta)} = \frac{1/2}{-\sqrt{5}/2} = \boxed{-\frac{1}{\sqrt{5}}}$$

(2) (a) [5 pts] Solve for x if

$$2^{x+3} = 4^{3x-1}$$

Solution:

Writing everything as a power of 2,

$$2^{x+3} = (2^2)^{3x-1} = 2^{2(3x-1)} = 2^{6x-2}$$

using exponent rules and expanding things out.
To have powers of the same base be equal, the exponents have to be the same. Therefore,

$$\begin{aligned} x + 3 &= 6x - 2 \\ \Rightarrow 5 &= 5x \\ \Rightarrow x &= 1 \end{aligned}$$

Thus, the answer is $\boxed{x = 1.}$

(b) [5 pts] Write the following quantity as a single logarithm:

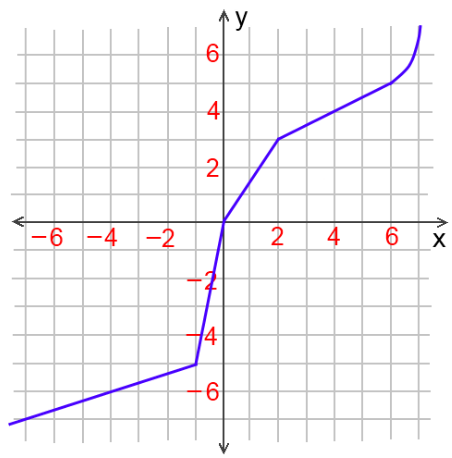
$$\ln(x) + \ln(y^2 - x) - 2\ln(z)$$

Solution:

Simplifying using logarithm rules,

$$\ln(x) + \ln(y^2 - x) - 2\ln(z) = \ln(x) + \ln(y^2 - x) - \ln(z^2) = \boxed{\ln\left(\frac{x(y^2 - x)}{z^2}\right)}$$

(3) The following is a graph of $f(x)$:



(a) [3 pts] Explain why $f(x)$ is one-to-one.

Solution:

$f(x)$ is one-to-one because it passes the horizontal line test.

(b) [3 pts] Find the value of $f^{-1}(3)$, explaining how you got it.

Solution:

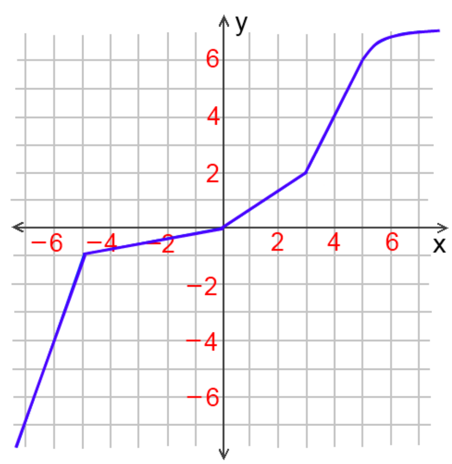
$f^{-1}(3)$ is the x-value corresponds to $y = 3$. Therefore,

$$\boxed{f^{-1}(3) = 2}$$

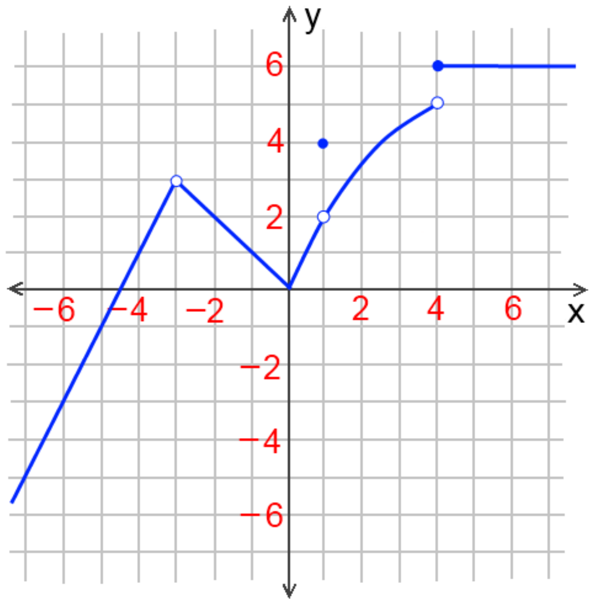
(c) [4 pts] Sketch the graph of $f^{-1}(x)$ on these axes:

Solution:

Reflecting along the line $y = x$, we get:



(4) Consider the $f(x)$ in the following graph:



(a) [5 pts] State whether $\lim_{x\to 1} f(x)$ exists, and calculate it if it does.

Solution:

$\lim_{x\to 1} f(x)$ exists: as the x-value approaches 1, the y-value approaches 2, and therefore

$$\boxed{\lim_{x\to 1} f(x) = 2}$$

(b) [5 pts] State whether $\lim_{x\to 4} f(x)$ exists, and calculate it if it does.

Solution:

$\lim_{x\to 4} f(x)$ doesn't exist, since the limit from the right is equal to 6, while the limit from the left is equal to 5.

(c) [5 pts] State all the points a for which $f(x)$ **NOT** continuous at a.

Solution:

The function has a jump discontinuity at 4 and removable discontinuities at -3 and 1. Therefore, the function is not continuous at

$$\boxed{x = -3, 1, 4}$$

(5) Calculate the following limits. You must show all your work to get credit. State if you're using continuity.

(a) [5 pts] $\lim_{x\to 1} \frac{x^2+1}{x+1}$

Solution:

The function $f(x) = \frac{x^2+1}{x+1}$ is a rational function, and as such is continuous on its domain. Since $x = 1$ doesn't make the denominator 0, $x = 1$ is in the domain of f. Therefore, $\lim_{x\to 1} f(x)$ can be evaluated by direct substitution, and is equal to $f(1)$. More concisely,

$$\boxed{\lim_{x\to 1} \frac{x^2+1}{x+1} = \frac{1^2+1}{1+1} = 1}$$

(b) [5 pts] $\lim_{x\to 0} \frac{\sqrt{3x+4}-2}{x}$

Solution:

Here, direct substitution results in $\frac{0}{0}$, which means that $x = 0$ is not in the domain of the function. Therefore, we need to do some simplifying calculations – we use the difference of squares formula after multiplying both top and bottom by the conjugate of the top:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{3x+4}-2}{x} &= \lim_{x\to 0} \frac{\sqrt{3x+4}-2}{x} \cdot \frac{\sqrt{3x+4}+2}{\sqrt{3x+4}+2} \\
&= \lim_{x\to 0} \frac{(\sqrt{3x+4})^2-2^2}{x(\sqrt{3x+4}+2)} = \lim_{x\to 0} \frac{3x+4-4}{x(\sqrt{3x+4}+2)} \\
&= \lim_{x\to 0} \frac{3x}{x(\sqrt{3x+4}+2)} \\
&= \lim_{x\to 0} \frac{3}{\sqrt{3x+4}+2}
\end{aligned}$$

canceling out the x in the top and bottom in the last step. We're now at the point where we can do direct substitution, since the function $f(x) = \frac{3}{\sqrt{3x+4}+2}$ is continuous on its domain, and $x = 0$ is in its domain. Therefore,

$$\lim_{x\to 0} \frac{3}{\sqrt{3x+4}+2} = \lim_{x\to 0} \frac{3}{\sqrt{3\cdot 0+4}+2} = \frac{3}{\sqrt{4}+2} = \frac{3}{4}$$

and therefore,

$$\boxed{\lim_{x\to 0} \frac{\sqrt{3x+4}-2}{x} = \frac{3}{4}}$$

(6) Let $f(x) = \dfrac{x^3 - 1}{x^2(x-3)}$.

(a) [5 pts] Calculate the horizontal asymptotes of $f(x)$. Show all your work to get credit.

Solution:

To calculate the horizontal asymptotes, calculate the limits as $x \to \infty$ and the limits as $x \to -\infty$. To assist in limit-taking, first divide both top and bottom by the highest power in the denominator, which is x^3 (this is the highest power when expanded out, NOT factored.)

$$\lim_{x\to\infty} \frac{x^3-1}{x^2(x-3)} = \lim_{x\to\infty} \frac{(x^3-1)/x^3}{x^2(x-3)/x^3} = \lim_{x\to\infty} \frac{1-1/x^3}{1-3/x}$$

As $x \to \infty$, both $\frac{1}{x^3}$ and $\frac{3}{x}$ approach 0. Therefore, plugging in 0 for them, we get

$$\lim_{x\to\infty} \frac{1-1/x^3}{1-3/x} = \frac{1-0}{1-0} = 1$$

Similarly,

$$\lim_{x\to-\infty} \frac{x^3-1}{x^2(x-3)} = \lim_{x\to-\infty} \frac{(x^3-1)/x^3}{x^2(x-3)/x^3} = \lim_{x\to-\infty} \frac{1-1/x^3}{1-3/x} = 1$$

since $\frac{1}{x^3}$ and $\frac{3}{x}$ also go to 0 as $x \to -\infty$. Therefore, there is the same asymptote at both ∞ and $-\infty$, and it is

$$\boxed{y = 1}$$

(b) [5 pts] Calculate the vertical asymptotes of $f(x)$.

Solution:

To calculate the vertical asymptotes, set the denominator to 0, and make sure that at those values the numerator is not 0 (if both are 0, then the expression $\frac{0}{0}$ is indeterminate instead of being $\pm\infty$.)
Setting the denominator equal to 0:

$$x^2(x-3) = 0 \Rightarrow x^2 = 0 \text{ or } x - 3 = 0$$
$$\Rightarrow x = 0 \text{ or } x = 3$$

Checking, at $x = 3$ the numerator is 26, while at $x = 0$ the numerator is -1. Therefore, they both work, so the vertical asymptotes are

$$\boxed{x = 0 \text{ and } x = 3}$$

(c) [10 pts] For each vertical asymptote $x = a$, calculate the one-sided limits $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$. For which of the vertical asymptotes does the two-sided limit $\lim_{x\to a} f(x)$ exist? Show all your work!

Solution:

Doing the calculations like we did in class, with the vertical asymptotes $x = 0$ and $x = 3$:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x^3-1}{x^2(x-3)} = \frac{(0^+)^3-1}{(0^+)^2(0^+-3)} = \frac{-1}{0^+(-3)} = \frac{-1}{0^-} = \infty$$
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x^3-1}{x^2(x-3)} = \frac{(0^-)^3-1}{(0^-)^2(0^--3)} = \frac{-1}{0^+(-3)} = \frac{-1}{0^-} = \infty$$

using the fact that $(0^-)^2 = (0^+)^2 = 0^+$ and rules such as the fact that a negative number divided by a negative number is positive. Therefore,

$$\lim_{x\to 0} f(x) = \infty$$

Similarly,

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{x^3-1}{x^2(x-3)} = \frac{(3^+)^3-1}{(3^+)^2(3^+-3)} = \frac{26}{9(0^+)} = \frac{26}{0^+} = \infty$$
$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x^3-1}{x^2(x-3)} = \frac{(3^-)^3-1}{(3^-)^2(3^--3)} = \frac{26}{9(0^-)} = \frac{26}{0^-} = -\infty$$

Therefore,

$$\lim_{x\to 3} f(x) \text{ DNE (does not exist)}$$

(7) Let $f(x)$ be a function satisfying $1 \leq f(x) \leq x^2 + 1$ on the interval $[-2, 2]$.

(a) [6 pts] Use the Squeeze Theorem to calculate $\lim_{x \to 0} f(x)$. Show all your work.

Solution:

Here, we have that
$$\lim_{x \to 0} 1 = 1 = \lim_{x \to 0}(x^2 + 1)$$
Therefore, by the Squeeze Theorem,
$$\boxed{\lim_{x \to 0} f(x) = 1}$$

(b) [4 pts] Could we use the Squeeze Theorem to calculate $\lim_{x \to 1} f(x)$? Why or why not?

Solution:

Here, we have that
$$\lim_{x \to 1} 1 = 1 \neq 2 = \lim_{x \to 1}(x^2 + 1)$$
Since the Squeeze Theorem can only be used when the two limits are equal, it can't be used here to calculate the limit of $f(x)$ as x approaches 1.

(8) (a) [6 pts] Use the Intermediate Value Theorem to justify why $x^2 = \cos(x)$ has a solution on the interval $(0, \pi/2)$.

Solution:

Rearranging the equation, we're solving for x such that

$$x^2 - \cos(x) = 0$$

Let $f(x) = x^2 - \cos(x)$. This function is continuous everywhere, since it's the difference of the two continuous functions x^2 and $\cos(x)$, and therefore it's continuous on the closed interval $[0, \pi/2]$. Now, we have that

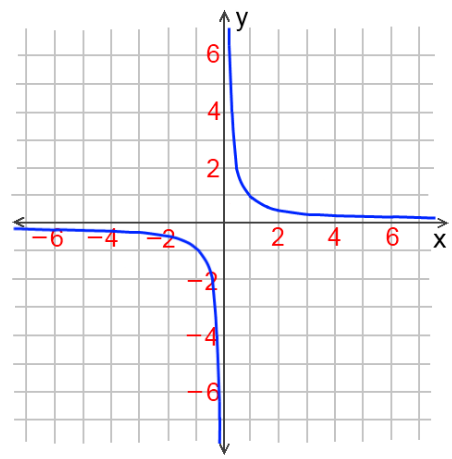
$$f(0) = 0^2 - \cos(0) = -1 < 0$$
$$f(\pi/2) = (\pi/2)^2 - \cos(\pi/2) = \pi^2/4 > 0$$

Therefore, 0 is between $f(0)$ and $f(\pi/2)$, so the Intermediate Value Theorem states that there exists a c on the interval $(0, \pi/2)$ such that $f(c) = 0$. Thus, c is a solution to $x^2 = \cos(x)$ on the interval $(0, \pi/2)$, so a solution has been shown to exist.

(b) [4 pts] On the right is the (hopefully familiar) graph of $f(x) = 1/x$. As you can see, $f(-1) = -1$ and $f(1) = 1$. However, there does **not** exist a c on the interval $(-1, 1)$ such that $f(c) = 0$. (Put another way, the graph doesn't cross the x-axis.) Explain why this does not contradict the Intermediate Value Theorem.

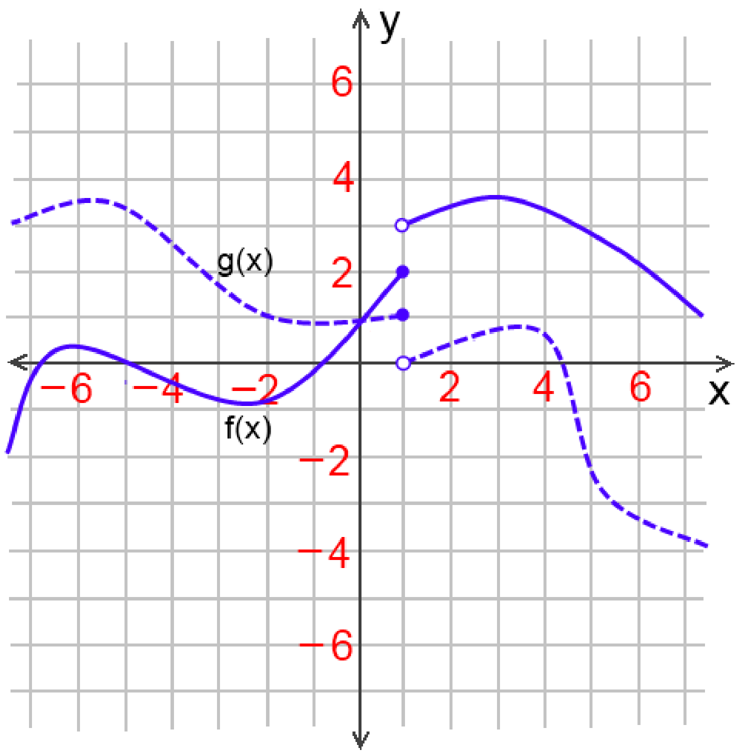
Solution:

The function $f(x) = 1/x$ is not continuous at 0, so it is not continuous on the interval $[-1, 1]$. Therefore, the Intermediate Value Theorem doesn't apply, and there's no contradiction.



(9) BONUS: [10 pts] Sketch two functions $f(x)$ and $g(x)$ such that neither of the limits $\lim_{x\to 1} f(x)$ and $\lim_{x\to 1} g(x)$ exist, but $\lim_{x\to 1}(f(x)+g(x))$ does exist. Explain why that is the case for the functions you chose.

Solution:



For the functions $f(x)$ and $g(x)$ in the picture above (where $f(x)$ is represented by the solid line, and $g(x)$ by the dotted line), the limit at 1 clearly doesn't exist. However, the limits from the left and right do exist, and therefore we can do the following calculation:

$$\lim_{x\to 1^-}(f(x)+g(x)) = \lim_{x\to 1^-} f(x) + \lim_{x\to 1^-} g(x) = 2+1 = 3$$
$$\lim_{x\to 1^+}(f(x)+g(x)) = \lim_{x\to 1^+} f(x) + \lim_{x\to 1^+} g(x) = 3+0 = 3$$

Thus, since $\lim_{x\to 1^-}(f(x)+g(x)) = \lim_{x\to 1^+}(f(x)+g(x)) = 3$, we see that

$$\lim_{x\to 1}(f(x)+g(x)) = 3$$

so the limit of $f(x)+g(x)$ as x approaches 1 does exist, just as we wanted.