# MATH 408N MIDTERM 2

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TA session: ____________________

**Show your work for all the problems. Good luck!**

(1) Use the limit definition of the derivative for the following questions. *You will get no points for using the rules learned later!*

(a) [5 pts] Find $f'(x)$ if $f(x) = 2x^2 + x$.

**Solution:**

Using the definition, we have that

$$f'(x) = \lim_{h \to 0} \frac{2(x+h)^2 + (x+h) - (2x^2 + x)}{h}.$$

Expanding the numerator and simplifying, we obtain

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 + x + h - 2x^2 - x}{h} \\ &= \lim_{h \to 0} \frac{4xh + 2h^2 + h}{h} = \lim_{h \to 0} 4x + 2h + 1 \\ &= \boxed{4x + 1} \end{aligned}$$

(b) [5 pts] Find $f'(4)$ if $f(x) = \frac{1}{x}$.

**Solution:**

Using the definition, we have that

$$f'(4) = \lim_{h \to 0} \frac{\frac{1}{4+h} - \frac{1}{4}}{h}.$$

Combining in the numerator, we have

$$f'(4) = \lim_{h \to 0} \frac{\frac{4}{4(4+h)} - \frac{4+h}{4(4+h)}}{h} = \lim_{h \to 0} = \lim_{h \to 0} \frac{\frac{-h}{4(4+h)}}{h} = \lim_{h \to 0} \frac{-1}{4(4+h)} = \boxed{-\frac{1}{16}}$$

(2) Differentiate the following functions, using whatever methods you think are best (you are now allowed to use all the rules). **You do NOT need to simplify your answer!**

(a) [5 pts] $f(x) = x^2 - 10x + 3\sqrt{x}$

**Solution:**

Rewriting as $f(x) = x^2 - 10x + 3x^{\frac{1}{2}}$, repeated use of the power rule tells us that

$$\boxed{f'(x) = 2x - 10 + \frac{3}{2}x^{-\frac{1}{2}}}$$

(b) [5 pts] $f(x) = \dfrac{\ln(x)}{x^2+1}$

**Solution:**

The quotient rule yields

$$\begin{aligned} f'(x) &= \frac{(x^2+1)\ln(x)' - \ln(x)(x^2+1)'}{(x^2+1)^2} \\ &= \boxed{\frac{(x^2+1)(\frac{1}{x}) - \ln(x)(2x)}{(x^2+1)^2}} \end{aligned}$$

(c) [5 pts] $f(x) = e^x \cos(x) + \arcsin(x)$

**Solution:**

Using the product rule, we find

$$\begin{aligned} f'(x) &= (e^x)' \cos(x) + e^x(\cos(x))' + \frac{1}{\sqrt{1-x^2}} \\ &= \boxed{e^x \cos(x) - e^x \sin(x) + \frac{1}{\sqrt{1-x^2}}} \end{aligned}$$

(3) Find the equations of the tangent lines in the following questions:
(a) [7 pts] $y = \tan(x^2) + 2x + 1$ at $x = 0$.

**Solution:**

Differentiating using the chain rule, we find

$$y' = 2x \cdot \sec^2(x^2) + 2.$$

When $x = 0$, we have that $y' = 2$ and $y = 1$. Therefore, the tangent line is given by the equation

$$y - 1 = 2(x - 0),$$

or

$$\boxed{y = 2x + 1}.$$

(b) [8 pts] $xy + e^{y-1} = x^2 - 1$ at $(2, 1)$.

**Solution:**

We use implicit differentiation, and find

$$y + xy' + e^{y-1}y' = 2x.$$

Solving for $y'$, we have

$$y'(x + e^{y-1}) = 2x - y,$$

or

$$y' = \frac{2x - y}{x + e^{y-1}}.$$

Therefore, at $(2, 1)$ we have

$$y' = \frac{2(2) - 1}{2 + e^{1-1}} = \frac{4 - 1}{2 + 1} = 1.$$

The tangent line is then given by

$$y - 1 = (1)(x - 2),$$

or

$$\boxed{y = x - 1}$$

(4) Solve the following differentiation problems:

(a) [5 pts] Find $f'(x)$ if $f(x) = \tan(x)^{x^2}$. Write your answer only in terms of $x$.

**Solution:**

Taking logarithms of both sides, we have

$$\ln f(x) = x^2 \ln(\tan(x))$$

Differentiating, using the product rule and the chain rule, we get

$$\begin{aligned} \frac{f'(x)}{f(x)} &= (x^2)' \ln(\tan(x)) + x^2(\ln(\tan(x)))' \\ &= 2x\ln(\tan(x)) + x^2\left(\frac{1}{\tan(x)}\right)(\tan(x))' \\ &= 2x\ln(\tan(x)) + x^2\left(\frac{1}{\tan(x)}\right)\sec^2(x) \end{aligned}$$

Multiplying it out, we have

$$\begin{aligned} f'(x) &= f(x)\left(2x\ln\tan(x) + \frac{x^2\sec^2(x)}{\tan(x)}\right) \\ &= \boxed{\tan(x)^{x^2}\left(2x\ln\tan(x) + \frac{x^2\sec^2(x)}{\tan(x)}\right)} \end{aligned}$$

(b) [5 pts] Find $F'(1)$ if $F(x) = f(g(x))$, $g(1) = 2, g'(1) = 3$, and $f'(2) = -1$.

**Solution:**

The chain rule implies that $F'(x) = f'(g(x))g'(x)$. Plugging in, we have

$$F'(1) = f'(g(1))g'(1) = f'(2)(3) = (-1)(3) = \boxed{-3}$$

(5) Find $y''$ for the $y$ defined in the following questions. **You do NOT need to simplify your answer!**

(a) [7 pts] $y = \ln(x^2 + \cos(x))$

**Solution:**

Differentiating, we have

$$\begin{aligned} y' &= \frac{1}{x^2+\cos(x)}(x^2+\cos(x))' = \frac{1}{x^2+\cos(x)}(2x-\sin(x)) \\ &= \frac{2x-\sin(x)}{x^2+\cos(x)} \end{aligned}$$

Using the quotient rule, we get the answer

$$\begin{aligned} y'' &= \frac{(x^2+\cos(x))(2x-\sin(x))' - (2x-\sin(x))(x^2+\cos(x))'}{(x^2+\cos(x))^2} \\ &= \boxed{\frac{(x^2+\cos(x))(2-\cos(x)) - (2x-\sin(x))(2x-\sin(x))}{(x^2+\cos(x))^2}} \end{aligned}$$

(b) [8 pts] $\sin(x+y) = 2y$ (Here, your answer can include both $x$ and $y$.)

**Solution:**

Using implicit differentiation, we get

$$\cos(x+y)(1+y') = 2y'.$$

Solving for $y'$, we get

$$y'(2-\cos(x+y)) = \cos(x+y),$$

or

$$y' = \frac{\cos(x+y)}{2-\cos(x+y)}$$

Now, differentiating using the quotient rule yields

$$\begin{aligned} y'' &= \left(\frac{\cos(x+y)}{2-\cos(x+y)}\right)' \\ &= \frac{(2-\cos(x+y))(\cos(x+y))' - (2-\cos(x+y))'\cos(x+y)}{(2-\cos(x+y))^2} \\ &= \frac{(2-\cos(x+y))(-\sin(x+y))(1+y') - \sin(x+y)(1+y')\cos(x+y)}{(2-\cos(x+y))^2} \end{aligned}$$

At this point, substituting the $y'$ we found earlier into the above equation *gives a completely correct answer*. Since this will take up more than the width of the page, I will simplify a bit before doing that:

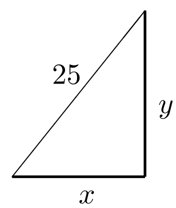
$$\begin{aligned} y'' &= \frac{(1+y')\left[(2-\cos(x+y))(-\sin(x+y)) - \sin(x+y)\cos(x+y)\right]}{(2-\cos(x+y))^2} \\ &= \frac{(1+y')(-2\sin(x+y))}{(2-\cos(x+y))^2} = \boxed{\frac{-2\left(1+\frac{\cos(x+y)}{2-\cos(x+y)}\right)\sin(x+y)}{(2-\cos(x+y))^2}} \end{aligned}$$

(6) A 25 foot ladder is sliding down a vertical wall. The top of the ladder is sliding down the wall at the rate of 2 ft/sec. (*You should express the answers to these questions as fractions: no need to make them decimals!*)

**Note:** This is the question from the practice! I did change the rate of sliding down the wall, but that's the only difference.

(a) [5 pts] How quickly is the bottom of the ladder moving away from the vertical wall when the top of the ladder is 20 feet away from the floor?

**Solution:**




We'll be using the picture as labelled above.

(i) *Given:* $y' = -2$ (we're given that it's sliding down at 1 foot/sec, and clearly $y$ is decreasing), $y = 20$.
*Find:* $x'$

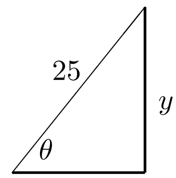
(ii) *Relationship:* Clearly, using Pythagoras, $x^2 + y^2 = 25^2 = 625$.

(iii) *Differentiate:* Differentiating both sides using chain rule,
$$2xx' + 2yy' = 0$$

(iv) *Solve for $x'$, plug in instantaneous info:* Here, we have that
$$2xx' = -2yy'$$
$$\Rightarrow x' = -\frac{2yy'}{2x} = -\frac{yy'}{x}$$
Now, we need to figure out the $x$ at the instant. Since $y = 20$ and $x^2 + y^2 = 625$, we have that
$$x^2 = 625 - y^2 = 625 - (20)^2 = 625 - 400 = 225$$
and therefore $x = \sqrt{225} = 15$. Therefore, plugging in:
$$x' = -\frac{20 \cdot (-2)}{15} = \frac{40}{15} = \frac{8}{3}$$
Thus, $\boxed{\text{the bottom is sliding at 8/3 ft/sec.}}$

(b) [5 pts] How quickly is the angle between the ladder and the floor (i.e. the horizontal) changing when the top of the ladder is 20 feet away from the floor? (To be precise, what is its speed?) Is it increasing or decreasing?

**Solution:**




(i) *Given:* $y' = -2$, $y = 20$.
*Find:* $\theta'$

(ii) *Relationship:* Since $\sin(\theta)$ is $\frac{\text{opp}}{\text{hyp}}$,

$$\sin(\theta) = \frac{y}{25}$$

(iii) *Differentiate:* Differentiating both sides using chain rule,

$$\cos(\theta)\theta' = \frac{y'}{25}$$

(iv) *Solve for $\theta'$, plug in instantaneous info:* Here, we have that

$$\theta' = \frac{\frac{y'}{25}}{\cos(\theta)} = \frac{y'}{25\cos(\theta)}$$

We need to figure out $\cos(\theta)$ at the instant. Now,

$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{625 - 400}}{25} = \frac{15}{25} = \frac{3}{5}$$

Plugging that in,

$$\theta' = \frac{(-2)}{25 \cdot 3/5} = -\frac{2}{15}$$

Therefore, **the angle is changing with the speed of 2/15 radians/sec.** Since the rate of change is negative, it is **decreasing.**

(7) Answer the following questions:

(a) [5 pts] Find the linearization $L(x)$ for $f(x) = \sqrt{x} + \dfrac{2}{\sqrt{x}}$ at $x = 4$.

**Solution:**

First, we compute that

$$f'(x) = \frac{1}{2}x^{-1/2} - x^{-3/2}.$$

We know that linearization at $x = 4$ is given by

$$L(x) = f(4) + f'(4)(x-4)$$

Calculating the values required in the above formula,

$$\begin{aligned} f(4) &= \sqrt{4} + \frac{2}{\sqrt{4}} = 2 + \frac{2}{2} = 2 + 1 = 3 \\ f'(4) &= \frac{1}{2}4^{-1/2} - 4^{-3/2} = \frac{1}{2} \cdot \frac{1}{4^{1/2}} - \frac{1}{4^{3/2}} \\ &= \frac{1}{2} \cdot \frac{1}{\sqrt{4}} - \frac{1}{(4^{1/2})^3} = \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{2^3} \\ &= \frac{1}{4} - \frac{1}{8} = \frac{2}{8} - \frac{1}{8} = \frac{1}{8} \end{aligned}$$

Plugging those values in, we get

$$\boxed{L(x) = 3 + \frac{1}{8}(x-4)}$$

You did *not* have to simplify further – indeed, the formula written above is easier to use in part (b) below!

(b) [5 pts] Use the result from part (a) to estimate $\sqrt{4.1} + \dfrac{2}{\sqrt{4.1}}$ as a fraction. ($0.1 = \frac{1}{10}$, if anyone needs to use that to get their fraction to look right. . . )

**Solution:**

For values close to 4, the above linearization is a good estimate of the function $f(x)$. Therefore, we can use the fact that $f(4.1) \approx L(4.1)$. Calculating,

$$\begin{aligned} L(4.1) &= 3 + \frac{1}{8}(4.1 - 4) = 3 + \frac{1}{8}(0.1) \\ &= 3 + \frac{1}{8} \cdot \frac{1}{10} = 3 + \frac{1}{80} \\ &= \boxed{3\frac{1}{80}} \end{aligned}$$

If you wanted to write it as a single fraction, you would get the estimate

$$L(4.1) = \boxed{\frac{241}{80}}$$

(8) BONUS: [10 pts] Let $f(x)$ be the following function:

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(That is, $f(x) = x^2 \sin(1/x)$ everywhere except at 0, and $f(0)$ is defined to be 0.) Use the limit definition of the derivative to show that $f'(0) = 0$.

**Solution:**

By one of our definitions, we have that

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Therefore, plugging in 0 for $a$, we get that

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x) - f(0)}{x}$$

By our definition of the function $f(x)$, we have that $f(0) = 0$. Furthermore, for all other $x$, we have that $f(x) = x^2 \sin(1/x)$. Plugging those in,

$$\begin{aligned} f'(0) &= \lim_{x \to 0} \frac{x^2 \sin(1/x) - 0}{x} = \lim_{x \to 0} \frac{x^2 \sin(1/x)}{x} \\ &= \lim_{x \to 0} x \sin(1/x) \end{aligned}$$

If you got this far in the question, you already got some points! A lot of people tried to use the formula $f(x) = x^2 \sin(1/x)$ to figure out $f(0)$, but that's not how the function is defined at 0, and it results in having the absurd quantity $1/0$ as part of the expression.

Now, the limit $\lim_{x \to 0} x \sin(1/x)$ is a pretty hard one, since it can't be split up into two pieces (do you see why not?) The key thing to notice is that $x$ approaches 0, while $\sin(1/x)$ is never bigger than 1 or smaller than $-1$. Therefore, $x \sin(1/x)$ is the product of a tiny number with a number that oscillates between $-1$ and 1. This should intuitively approach 0... it should also make you think of the Squeeze Theorem! Accordingly, note that

$$-1 \leq \sin(1/x) \leq 1$$

which can be manipulated to show that

$$-|x| \leq x \sin(1/x) \leq |x|$$

Finally, since

$$\lim_{x \to 0} (-|x|) = 0 = \lim_{x \to 0} |x|$$

the Squeeze Theorem tells us that

$$\lim_{x \to 0} x \sin(1/x) = 0$$

which is precisely what we wanted!

**Note:** This bonus was really hard! No one tried to use the Squeeze Theorem at all. I expect the bonus on the final will be considerably easier.