MATH 408N PRACTICE MIDTERM 2

Name:__________________

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Bormashenko

TA session: __________________

Show your work for all the problems. Good luck!

(1) Use the limit definition of the derivative for the following questions. *You will get no points for using the rules learned later!*

(a) [5 pts] Find $f'(1)$ if $f(x) = \sqrt{x}$.

Solution:

By definition,

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\sqrt{1+h} - 1}{h}$$

Now, multiplying top and bottom by the conjugate, then using difference of squares:

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{\sqrt{1+h} - 1}{h} \cdot \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1} \\ &= \lim_{h\to 0} \frac{(\sqrt{1+h})^2 - 1^2}{h(\sqrt{1+h} + 1)} = \lim_{h\to 0} \frac{h}{h(\sqrt{1+h} + 1)} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{1+h} + 1} \end{aligned}$$

Finally, plugging in $h = 0$ doesn't result in $\frac{0}{0}$. Therefore, we plug in, getting that

$$f'(1) = \frac{1}{\sqrt{1} + 1} = \boxed{\frac{1}{2}}$$

Note: You could also use the definition

$$f'(1) = \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1}$$

and use a similar trick.

(b) [5 pts] Find $f'(x)$ if $f(x) = x^2 + x + 1$.

Solution:

By definition,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 + (x+h) + 1 - (x^2 + x + 1)}{h}$$

Now, expanding things out,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + x + h + 1 - x^2 - x - 1}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2 + h}{h} = \lim_{h\to 0} \frac{h(2x + h + 1)}{h} \\ &= \lim_{h\to 0} (2x + h + 1) = \boxed{2x + 1} \end{aligned}$$

(2) Differentiate the following functions, using whatever methods you think are best (you are now allowed to use all the rules):

(a) [5 pts] $f(x) = x^3 + 3x + 5$

Solution:

Using the sum rules and the fact that $(x^n)' = nx^{n-1}$,

$$\boxed{f'(x) = 3x^2 + 3}$$

(b) [5 pts] $f(x) = \arctan\left(x^2\right) e^x$

Solution:

By the product rule,

$$f'(x) = \left(\arctan\left(x^2\right)\right)' e^x + \arctan\left(x^2\right)\left(e^x\right)'$$

Using the chain rule,

$$\left(\arctan\left(x^2\right)\right)' = \frac{1}{1+(x^2)^2} \cdot (x^2)' = \frac{2x}{1+x^4}$$

Using straightforward differentiation rules,

$$(e^x)' = e^x$$

Now, plugging the expressions for $(e^x)'$ and $\left(\arctan\left(x^2\right)\right)'$ back into the $f'(x)$ formula,

$$f'(x) = \boxed{\frac{2x}{1+x^4}e^x + \arctan\left(x^2\right) e^x}$$

(c) [5 pts] $f(x) = \dfrac{2x\sin(x)}{\ln(x+1)}$

Solution:

Using the quotient rule,

$$f'(x) = \frac{(2x\sin(x))'\ln(x+1) - 2x\sin(x)(\ln(x+1))'}{(\ln(x+1))^2}$$

Now, working out the derivative appearing above separately:

$$(2x\sin(x))' = (2x)'\sin(x) + 2x(\sin(x))' = 2\sin(x) + 2x\cos(x)$$
$$\ln(x+1)' = \frac{1}{x+1}(x+1)' = \frac{1}{x+1}$$

where the first one uses product rule, and the second uses chain rule. Plugging those back in,

$$\boxed{f'(x) = \frac{(2\sin(x) + 2x\cos(x))\ln(x+1) - 2x\sin(x)\frac{1}{x+1}}{(\ln(x+1))^2}}$$

(You can expand this out a bit more, but you don't have to! Careful with the parentheses, though...)

(d) [5 pts] $f(x) = \sin(x)^{\cos(x)}$

Solution:

This one HAS to be done using logarithmic differentiation! Any attempts to use the power rule or the exponent rule will get a wrong answer, because there's a variable both in the exponent and in the base.
Therefore, proceeding by taking ln of both sides:

$$\ln(f(x)) = \ln\left(\sin(x)^{\cos(x)}\right) = \cos(x)\ln(\sin(x))$$

using the log rule which says that $\ln(x^r) = r\ln(x)$.
Differentiating both sides using chain rule and product rule:

$$\begin{aligned}
\ln(f(x))' = (\cos(x)\ln(\sin(x)))' &= \cos(x)'\ln(\sin(x)) + \cos(x)(\ln(\sin(x)))' \\
&= -\sin(x)\ln(\sin(x)) + \cos(x)\frac{1}{\sin(x)}(\sin(x))' \\
&= -\sin(x)\ln(\sin(x)) + \frac{\cos(x)}{\sin(x)}\cos(x) \\
&= -\sin(x)\ln(\sin(x)) + \frac{\cos^2(x)}{\sin(x)}
\end{aligned}$$

Now, using chain rule, $(\ln(f(x)))' = \frac{f'(x)}{f(x)}$. Therefore,

$$\begin{aligned}
\frac{f'(x)}{f(x)} &= -\sin(x)\ln(\sin(x)) + \frac{\cos^2(x)}{\sin(x)} \\
\Rightarrow f'(x) &= f(x)\left(-\sin(x)\ln(\sin(x)) + \frac{\cos^2(x)}{\sin(x)}\right) \\
&= \boxed{\sin(x)^{\cos(x)}\left(-\sin(x)\ln(\sin(x)) + \frac{\cos^2(x)}{\sin(x)}\right)}
\end{aligned}$$

(3) Find the equations of the tangent lines to the following graphs at the given points:

(a) [7 pts] $y = x^2 \ln(x) + 1$ at $(1, 1)$.

Solution:

Differentiating using product rule, after using the fact that $(1)' = 0$:

$$y' = (x^2 \ln(x))' + (1)' = (x^2)' \ln(x) + x^2 \ln(x)'$$
$$= 2x \ln(x) + x^2 \cdot \frac{1}{x} = 2x \ln x + x$$

Thus, $y'(1) = 2\ln(1) + 1 = 2 \cdot 0 + 1 = 1$. Therefore, our tangent line has slope 1 and contains the point $(1, 1)$. Using the point-slope formula, the equation of the line is

$$y - 1 = 1(x - 1) = x - 1$$

which simplifies to $\boxed{y = x}$.

(b) [8 pts] $x^2 + y^2 = xe^y$ at $(1, 0)$.

Solution:

Using implicit differentiation to differentiate both sides,

$$2x + 2yy' = x(e^y)' + (x)'e^y = xe^y y' + e^y$$

Now, moving all the y' to the same side:

$$2yy' - xe^y y' = e^y - 2x$$
$$y'(2y - xe^y) = e^y - 2x$$
$$\Rightarrow y' = \frac{e^y - 2x}{2y - xe^y}$$

Plugging in the point $(1, 0)$, we get that y' at $(1, 0)$ is

$$\frac{e^0 - 2}{2 \cdot 0 - 1 \cdot e^0} = \frac{1 - 2}{0 - 1} = \frac{-1}{-1} = 1$$

Therefore, the slope of the line is 1, while the point on the line is $(1, 0)$. Hence, using point-slope,

$$(y - 0) = 1(x - 1)$$

and simplifying, the equation of the line is $\boxed{y = x - 1}$.

(4) Let $f(x)$ and $g(x)$ satisfy $f(1) = 3, g(1) = 2, f'(1) = 1, g'(1) = -1$, and $f'(2) = -2$.

(a) [5 pts] If $F(x) = f(x)g(x)$, calculate $F'(1)$.

Solution:

Using the product rule,

$$F'(x) = f'(x)g(x) + f(x)g'(x)$$

Therefore,

$$F'(1) = f'(1)g(1) + f(1)g'(1) = 1 \cdot 2 + 3 \cdot (-1) = \boxed{-1}$$

(b) [5 pts] If $G(x) = f(g(x))$, calculate $G'(1)$.

Solution:

Using the chain rule,

$$G'(x) = f'(g(x))g'(x)$$

Therefore,

$$G'(1) = f'(g(1))g'(1) = f'(2)g'(1) = (-2)(-1) = \boxed{2}$$

(c) [5 pts] If $H(x) = g(f(x))$, do we have enough information to calculate $H'(1)$?

Solution:

Again using the chain rule,

$$H'(x) = g'(f(x))f'(x)$$

Thus,

$$H'(1) = g'(f(1))f'(1) = g'(3)f'(1) = g'(3)$$

Since we don't know $g'(3)$, we $\boxed{\text{don't have enough information.}}$

(5) For the following graphs, find all the points at which the tangent line to the graph is parallel to $y = 2x + 3$. *Find both coordinates of each point for full marks!*

(a) [5 pts] $y = x^2 + 2x + 3$.

Solution:

Two lines are parallel if they have the same slope. Since the slope of $y = 2x + 3$ is 2, we're looking for points at which $y' = 2$.
Differentiating the function, we get

$$y' = 2x + 2$$

Setting it equal to 2 gets

$$\begin{aligned} 2 &= 2x + 2 \\ \Rightarrow 0 &= 2x \end{aligned}$$

and therefore, $x = 0$. Since $y = x^2 + 2x + 3$, the y-coordinate of this point is $0^2 + 2 \cdot 0 + 3 = 3$, so the only point at which the tangent is parallel to $y = 2x + 3$ is $\boxed{(0, 3)}$.

(b) [5 pts] $x^2 + y^2 = 5$.

Solution:

LIke in part (a), we set y' to 2. To find y', we need to use implicit differentiation. Accordingly, differentiating both sides:

$$\begin{aligned} 2x + 2yy' &= 0 \\ \Rightarrow 2yy' &= -2x \\ \Rightarrow y' &= \frac{-2x}{2y} = -\frac{x}{y} \end{aligned}$$

Therefore, at a point where $y' = 2$, we have that

$$-\frac{x}{y} = 2$$

so $x = -2y$. Plugging that back into the original equation,

$$\begin{aligned} (-2y)^2 + y^2 &= 5 \\ \Rightarrow 4y^2 + y^2 &= 5 \\ \Rightarrow 5y^2 &= 5 \\ \Rightarrow y^2 &= 1 \\ \Rightarrow y &= \pm\sqrt{1} = \pm 1 \end{aligned}$$

Thus, $y = 1$ or $y = -1$. Since we have that $x = -2y$, this results in the two points $\boxed{(-2, 1) \text{ and } (2, -1)}$.

(6) Find y'' for the following graphs:

(a) [7 pts] $y = \sin(\cos(x))$

Solution:

Since y'' is the derivative of y', we first find y'. Using the chain rule,

$$\begin{aligned} y' &= \cos(\cos(x)) \cdot (\cos(x))' = \cos(\cos(x))(-\sin(x)) \\ &= -\sin(x)\cos(\cos(x)) \end{aligned}$$

To find y'', we differentiate again. This time, we first use product rule:

$$\begin{aligned} y'' &= (-\sin(x)\cos(\cos(x)))' = -(\sin(x)'\cos(\cos(x)) + \sin(x)\cos(\cos(x))') \\ &= -\cos(x)\cos(\cos(x)) - \sin(x)\cos(\cos(x))' \end{aligned}$$

Finally, using the chain rule:

$$\cos(\cos(x))' = -\sin(\cos(x))(\cos(x))' = -\sin(\cos(x))(-\sin(x)) = \sin(x)\sin(\cos(x))$$

Plugging that back in,

$$\begin{aligned} y'' &= -\cos(x)\cos(\cos(x)) - \sin(x)(\sin(x)\sin(\cos(x))) \\ &= \boxed{-\cos(x)\cos(\cos(x)) - \sin^2(x)\sin(\cos(x))} \end{aligned}$$

(b) [8 pts] $x^2 + xy + y^2 = 2$ (Here, your answer can include both x and y.)

Solution:

Here, we use implicit differentiation to find y'. Differentiating both sides:

$$\begin{aligned} (x^2 + xy + y^2)' &= (2)' \\ \Rightarrow 2x + xy' + y + 2yy' &= 0 \end{aligned}$$

using product rule (and the fact that $x' = 1$ for xy and chain rule for y^2.)
Now, putting all the y' on one side, and all the other terms on the other side:

$$\begin{aligned} xy' + 2yy' &= -2x - y \\ \Rightarrow (x + 2y)y' &= -2x - y \\ \Rightarrow y' &= \frac{-2x - y}{x + 2y} = -\frac{2x + y}{x + 2y} \end{aligned}$$

Differentiating y' again using quotient rule,

$$\begin{aligned} y'' &= \left(-\frac{2x + y}{x + 2y}\right)' = -\frac{(x + 2y)(2x + y)' - (x + 2y)'(2x + y)}{(x + 2y)^2} \\ &= -\frac{(x + 2y)(2 + y') - (1 + 2y')(2x + y)}{(x + 2y)^2} \end{aligned}$$

At this stage, you will get **FULL MARKS** if you plug in the expression for y' from earlier and leave it unsimplified in terms of x and y. For aesthetic reasons, I will keep simplifying:

$$\begin{aligned} y'' &= -\frac{2x + 4y + xy' + 2yy' - (2x + 4xy' + y + 2yy')}{(x + 2y)^2} \\ &= -\frac{3y - 3xy'}{(x + 2y)^2} = \frac{3xy' - 3y}{(x + 2y)^2} \end{aligned}$$

Here, I will finally plug in the expression for y' from before, which was $y' = -\frac{2x+y}{x+2y}$:

$$y'' = \frac{3x(-\frac{2x+y}{x+2y}) - 3y}{(x+2y)^2} = \frac{3x(-\frac{2x+y}{x+2y}) - 3y}{(x+2y)^2} \times \frac{x+2y}{x+2y}$$
$$= -\frac{-3x(2x+y) - 3y(x+2y)}{(x+2y)^3} = \boxed{-\frac{6x^2+6xy+6y^2}{(x+2y)^3}}$$

And if you want to be really fancy... notice that $6x^2+6xy+6y^2 = 6(x^2+xy+y^2) = 6 \cdot 2 = 12$ since the original equation was $x^2+xy+y^2 = 2$. Therefore,

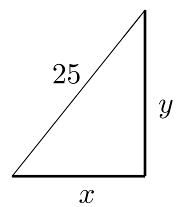
$$\boxed{y'' = -\frac{12}{(x+2y)^3}}$$

But I'm not Quest – you don't need to do all this! It's presented here entirely as a guide on simplifying.

(7) A 25 foot ladder is sliding down a vertical wall. The top of the ladder is sliding down the wall at the rate of 1 ft/sec. (*You should express the answers to these questions as fractions: no need to make them decimals!*)

(a) [7 pts] How quickly is the bottom of the ladder moving away from the vertical wall when the top of the ladder is 20 feet away from the floor?

Solution:



We'll be using the picture as labelled above.

(i) *Given:* $y' = -1$ (we're given that it's sliding down at 1 foot/sec, and clearly y is decreasing), $y = 20$.
Find: x'

(ii) *Relationship:* Clearly, using Pythagoras, $x^2 + y^2 = 25^2 = 625$.

(iii) *Differentiate:* Differentiating both sides using chain rule,

$$2xx' + 2yy' = 0$$

(iv) *Solve for x', plug in instantaneous info:* Here, we have that

$$2xx' = -2yy'$$
$$\Rightarrow x' = -\frac{2yy'}{2x} = -\frac{yy'}{x}$$

Now, we need to figure out the x at the instant. Since $y = 20$ and $x^2 + y^2 = 625$, we have that

$$x^2 = 625 - y^2 = 625 - (20)^2 = 625 - 400 = 225$$

and therefore $x = \sqrt{225} = 15$. Therefore, plugging in:

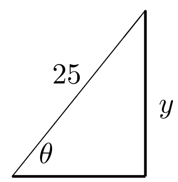
$$x' = -\frac{20 \cdot (-1)}{15} = \frac{20}{15} = \frac{4}{3}$$

Thus, $\boxed{\text{the bottom is sliding at } 4/3 \text{ ft/sec.}}$

(b) [8 pts] How quickly is the angle between the ladder and the floor (i.e. the horizontal) changing when the top of the ladder is 20 feet away from the floor? Is it increasing or decreasing?

Note: I'm going to write down two solutions to this problem. The first is by far the easier one, but for some reason most people opted for the second one…

Solution 1:



We'll be using the picture as labelled above. This solution does NOT use the result of part (a), which of course is preferable.

(i) *Given:* $y' = -1$, $y = 20$.
Find: θ'

(ii) *Relationship:* Since $\sin(\theta)$ is $\frac{\text{opp}}{\text{hyp}}$,

$$\sin(\theta) = \frac{y}{25}$$

(iii) *Differentiate:* Differentiating both sides using chain rule,

$$\cos(\theta)\theta' = \frac{y'}{25}$$

(iv) *Solve for θ', plug in instantaneous info:* Here, we have that

$$\theta' = \frac{\frac{y'}{25}}{\cos(\theta)} = \frac{y'}{25\cos(\theta)}$$

We need to figure out $\cos(\theta)$ at the instant. Now,

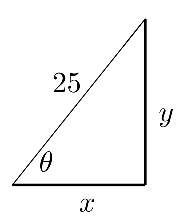
$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{625 - 400}}{25} = \frac{15}{25} = \frac{3}{5}$$

Plugging that in,

$$\theta' = \frac{(-1)}{25 \cdot 3/5} = -\frac{1}{15}$$

Therefore, $\boxed{\text{the angle is changing with the speed of } 1/15 \text{ radians/sec.}}$ Since the rate of change is negative, it is $\boxed{\text{decreasing.}}$

Solution 2:



We'll be using the picture as labelled above. This solution uses the results from part (a) as given.

(i) *Given:* $y' = -1$, $x' = \frac{4}{3}$, $y = 20$, $x = 15$
Find: θ'

(ii) *Relationship:* Since $\tan(\theta)$ is $\frac{\text{opp}}{\text{adj}}$,

$$\tan(\theta) = \frac{y}{x}$$

(iii) *Differentiate:* Differentiating both sides using chain rule, and quotient rule on the right-hand side:

$$\sec^2(\theta)\theta' = \frac{xy' - x'y}{x^2}$$

(iv) *Solve for θ', plug in instantaneous info:* Here, we have that

$$\theta' = \frac{\frac{xy'-x'y}{x^2}}{\sec^2(\theta)} = \frac{xy'-x'y}{x^2\sec^2(\theta)}$$

We need to figure out $\sec(\theta)$ at the instant. Now,

$$\sec(\theta) = \frac{\text{hyp}}{\text{adj}} = \frac{25}{x} = \frac{25}{15} = \frac{5}{3}$$

Plugging that in, along with everything else,

$$\theta' = \frac{15(-1)-(4/3)20}{15^2(5/3)^2} = \frac{-45/3-80/3}{225\cdot 25/9}$$
$$= -\frac{125/3}{625} = -\frac{125}{3\cdot 625} = -\frac{1}{15}$$

Same answer! So again, $\boxed{\text{the angle is changing with the speed of } 1/15 \text{ radians/sec.}}$ Since the rate of change is negative, it is $\boxed{\text{decreasing.}}$

(8) Answer the following questions:

(a) [5 pts] Find the linearization $L(x)$ to $f(x) = \frac{1}{\sqrt{x+1}}$ at $x = 3$.

Solution:

The linearization of $f(x)$ at $x = a$ is

$$L(x) = f(a) + f'(a)(x - a)$$

This formula uses the derivative of f, so let's first find $f'(x)$ using the chain rule:

$$f'(x) = \frac{1}{\sqrt{x+1}} = \left((x+1)^{-1/2}\right)' = -\frac{1}{2}(x+1)^{-3/2}(x+1)' = -\frac{1}{2}(x+1)^{-3/2}$$

Now, since we're finding the linearization at 3, and therefore $a = 3$, we have that

$$\begin{aligned} f(a) = f(3) &= \frac{1}{\sqrt{3+1}} = \frac{1}{\sqrt{4}} = \frac{1}{2} \\ f'(a) = f'(3) &= -\frac{1}{2}(3+1)^{-3/2} = -\frac{1}{2}(4^{1/2})^{-3} \\ &= -\frac{1}{2}2^{-3} = -\frac{1}{2} \cdot \frac{1}{2^3} = -\frac{1}{16} \end{aligned}$$

Plugging these values into the formula,

$$\begin{aligned} L(x) &= \frac{1}{2} - \frac{1}{16}(x - 3) = \frac{1}{2} - \frac{x}{16} + \frac{3}{16} \\ &= -\frac{x}{16} + \frac{8}{16} + \frac{3}{16} = \boxed{-\frac{x}{16} + \frac{11}{16}} \end{aligned}$$

(b) [5 pts] Use linearization to provide an estimate for $\sqrt[3]{29}$, given that $\sqrt[3]{27} = 3$.

Solution:

Since we're estimating a cube root, we need to find a linearization of $f(x) = \sqrt[3]{x}$. Since the estimate is for x close to 27, the linearization is at $a = 27$. As above,

$$L(x) = f(a) + f'(a)(x - a)$$

Calculating,

$$f'(x) = (\sqrt[3]{x})' = \left(x^{1/3}\right)' = \frac{1}{3}x^{-2/3}$$

Therefore,

$$\begin{aligned} f(a) &= \sqrt[3]{a} = \sqrt[3]{27} = 3 \\ f'(a) &= \frac{1}{3}(27)^{-2/3} = \frac{1}{3}(27^{1/3})^{-2} = \frac{1}{3}3^{-2} \\ &= \frac{1}{3} \cdot \frac{1}{9} = \frac{1}{27} \end{aligned}$$

Plugging those values in,

$$L(x) = 3 + \frac{1}{27}(x - 27)$$

We leave $L(x)$ in this form because it makes the next step easier. We need to find an estimate for $f(29)$, and as usual we use $L(29)$. Therefore, an estimate for $\sqrt[3]{29}$ is

$$L(29) = 3 + \frac{1}{27}(29 - 27) = 3 + \frac{2}{27} = \boxed{3\frac{2}{27}}$$