# MATH 408N PRACTICE FINAL

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**Show your work for all the problems. Good luck!**

(1) Let $f(x) = \frac{e^x}{e^x - 1}$.

(a) [5 pts] State the domain and range of $f(x)$.

**Solution:**

Since $e^x$ is defined for all $x$, we see that $f(x)$ is defined as long as the denominator is not 0. The denominator is 0 if

$$e^x - 1 = 0 \Rightarrow e^x = 1$$
$$\Rightarrow x = \ln(1) = 0$$

Thus, the domain is all $x \neq 0$, or in other words, $\boxed{(-\infty, 0) \cup (0, \infty).}$

To figure out the range, let's write

$$f(x) = \frac{e^x}{e^x - 1} = 1 + \frac{1}{e^x - 1}$$

Since $e^x$ can take on any positive value, $e^x - 1$ can take on any value greater than $-1$. Therefore, $\frac{1}{e^x-1}$ takes on values in $(0, \infty)$ and in $(-\infty, -1)$. Since we add 1 to this quantity, the range of $f$ is $\boxed{(-\infty, 0) \cup (1, \infty)}$

(b) [5 pts] Calculate a formula for $f^{-1}(x)$.

**Solution:**

We take the equation $y = f(x)$, solve for $x$, then swap $x$ and $y$ to find $f^{-1}(x)$.

$$y = \frac{e^x}{e^x - 1} \Rightarrow y(e^x - 1) = e^x$$
$$\Rightarrow ye^x - y = e^x$$
$$\Rightarrow e^x(y - 1) = y$$
$$\Rightarrow e^x = \frac{y}{y - 1}$$

Taking ln of both sides, we get that $x = \ln\left(\frac{y}{y-1}\right)$. Finally, swapping $x$ and $y$ we see that

$$\boxed{f^{-1}(x) = y = \ln\left(\frac{x}{x - 1}\right)}$$

(c) [5 pts] Find the domain and range of $f^{-1}$.

**Solution:**

We know that the domain of $f^{-1}$ is the range of $f$, and the range of $f^{-1}$ is the domain of $f$. Thus, the domain of $f^{-1}$ is $\boxed{(-\infty, 0) \cup (1, \infty)}$ and the range of $f^{-1}$ is $\boxed{(-\infty, 0) \cup (0, \infty).}$

(2) Calculate the following limits, using whatever tools are appropriate. State which results you're using for each question.

(a) [5 pts] $\lim_{x\to 1} \frac{x^2+1}{x-2}$

**Solution:**

Since $\frac{x^2+1}{x-2}$ is a rational function (a quotient of polynomials), it's continuous everywhere on its domain. Since 1 is in its domain, we can calculate the limit by plugging in. Therefore,

$$\lim_{x\to 1} \frac{x^2+1}{x-2} = \frac{1^2+1}{1-2} = \boxed{-2}$$

(b) [5 pts] $\lim_{x\to 0} \frac{\frac{1}{x} - \frac{1}{x-1}}{x^{-1}}$

**Solution:**

Here, we can't just plug in because we get expressions like 1/0. Therefore, doing a bit of algebra,

$$\begin{aligned}\lim_{x\to 0} \frac{\frac{1}{x} - \frac{1}{x-1}}{x^{-1}} &= \lim_{x\to 0} \frac{\frac{x-1}{x(x-1)} - \frac{x}{x(x-1)}}{x^{-1}} \\ &= \lim_{x\to 0} \frac{-\frac{1}{x(x-1)}}{x^{-1}} = -\lim_{x\to 0} \frac{1}{x^{-1}x(x-1)} \\ &= -\lim_{x\to 0} \frac{1}{x-1}\end{aligned}$$

Since $\frac{1}{x-1}$ is continuous at 0, this limit can be evaluated by plugging. Therefore,

$$\lim_{x\to 0} \frac{\frac{1}{x} - \frac{1}{x-1}}{x^{-1}} = -\frac{1}{0-1} = \boxed{1}$$

(c) [5 pts] $\lim_{x\to 0} \frac{\cos(x)-1}{e^x - x - 1}$

**Solution:**

This is a limit of the form $\frac{0}{0}$. Therefore, using L'Hospital's rule,

$$\lim_{x\to 0} \frac{\cos(x)-1}{e^x - x - 1} = \lim_{x\to 0} \frac{(\cos(x)-1)'}{(e^x - x - 1)'} = \lim_{x\to 0} \frac{-\sin(x)}{e^x - 1}$$

This is still of the form $\frac{0}{0}$, so we can use L'Hospital's again:

$$\lim_{x\to 0} \frac{-\sin(x)}{e^x - 1} = \lim_{x\to 0} \frac{-\cos(x)}{e^x}$$

Finally, $\frac{-\cos(x)}{e^x}$ is continuous at 0, so we can plug in. Therefore,

$$\lim_{x\to 0} \frac{\cos(x)-1}{e^x - x - 1} = \frac{-\cos(0)}{e^0} = \boxed{-1}$$

(d) [5 pts] $\lim_{x\to 2} f(x)$, where $2 \le f(x) \le x^2 - 2$ for all $x \in [1,4]$.

**Solution:**

This uses the Squeeze Theorem. Here, we have that

$$\lim_{x\to 2} 2 = 2 = \lim_{x\to 2}(x^2 - 2)$$

Therefore, since $f(x)$ is between 2 and $x^2 - 2$ on $[1,4]$, we also have that

$$\lim_{x\to 2} f(x) = \boxed{2}$$

(e) [5 pts] $\lim_{x\to\infty} \left(1 - \frac{1}{x}\right)^{3x+1}$

**Solution:**

This is an interderminate of the form $1^\infty$. Let $L = \lim_{x\to\infty}\left(1 - \frac{1}{x}\right)^{3x+1}$. Then, taking ln of both sides, we have that

$$\begin{aligned}\ln(L) &= \ln\left(\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^{3x+1}\right) = \lim_{x\to\infty} \ln\left(\left(1-\frac{1}{x}\right)^{3x+1}\right) \\ &= \lim_{x\to\infty}(3x+1)\ln\left(1-\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1-\frac{1}{x}\right)}{1/(3x+1)}\end{aligned}$$

This is now an indeterminate of the form $\frac{0}{0}$. Therefore, using L'Hospital's, and then doing some algebra:

$$\begin{aligned}\lim_{x\to\infty}\frac{\ln\left(1-\frac{1}{x}\right)}{1/(3x+1)} &= \lim_{x\to\infty}\frac{\frac{1}{1-1/x}\cdot\frac{1}{x^2}}{-3/(3x+1)^2} = \lim_{x\to\infty}\frac{(3x+1)^2}{-3(1-1/x)x^2} \\ &= \lim_{x\to\infty}\frac{9x^2+6x+1}{-3x^2+3x}\end{aligned}$$

Now, if you remember the rule for taking limits like these, you can read it off from the highest coefficients and see that it's going to be $\frac{9}{-3} = -3$. Doing the actual calculation we would need to write down by dividing top and bottom by $x^2$,

$$\begin{aligned}\lim_{x\to\infty}\frac{9x^2+6x+1}{-3x^2+3x} &= \lim_{x\to\infty}\frac{(9x^2+6x+1)/x^2}{(-3x^2+3x)/x^2} = \lim_{x\to\infty}\frac{9+6/x+1/x^2}{-3+3/x} \\ &= \frac{\lim_{x\to\infty}(9+6/x+1/x^2)}{\lim_{x\to\infty}(-3+3/x)} = \frac{9}{-3} = -3\end{aligned}$$

Since this was a calculation after taking the ln, we now know that $\ln(L) = -3$. Therefore,

$$\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^{3x+1} = L = e^{\ln(L)} = \boxed{e^{-3}}$$

(f) $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}$

**Solution:**

Using the standard difference of squares trick,

$$\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2} = \lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x\to 4}\frac{(x-4)(\sqrt{x}+2)}{x-4} = \lim_{x\to 4}\sqrt{x}+2$$

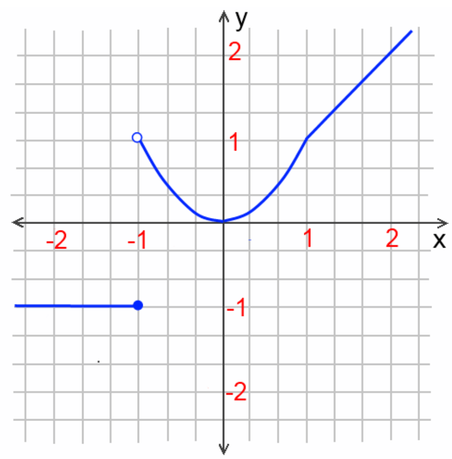
Since $\sqrt{x}+2$ is positive, we can just plug in to get the answer. Thus, $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2} = \sqrt{4}+2 = \boxed{4}$

(3) Let the function $f(x)$ be defined piecewise as follows:

$$f(x) = \begin{cases} -1 & x \leq -1 \\ x^2 & -1 < x < 1 \\ x & x \geq 1 \end{cases}$$

(a) [5 pts] Sketch a graph of this function.

**Solution:**




(b) [10 pts] State the intervals on which $f(x)$ is continuous. Do a limit calculation checking for continuity at any points where this is necessary.

**Solution:**

It should be clear from the above picture that $f(x)$ is continuous everywhere except at $-1$: that is, $f(x)$ is continuous on $\boxed{(-\infty, -1) \cup (-1, \infty)}$. However, we're asked to check this not only using the picture, but the limit definition. Recall that $f(x)$ is continuous at $a$ if

$$\lim_{x \to a} f(x) = f(a)$$

Since the functions $y = -1, y = x^2$, and $y = x$ are continuous everywhere, and these make up our three pieces, we know that $f(x)$ must be continuous eveywhere except the places where the functions 'connect.' Therefore, we only need to check whether $f(x)$ is continuous at $-1$ and 1. Now, note that

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} -1 = -1$$

using the fact that a little to the left of $-1$, $f(x) = -1$. Similarly,

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} x^2 = (-1)^2 = 1$$

Therefore,

$$\lim_{x \to -1^-} f(x) \neq \lim_{x \to -1^+} f(x)$$

so $\lim_{x \to -1} f(x)$ doesn't exist. Hence, $f(x)$ isn't continuous at $-1$.
Similarly, checking at 1:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1$$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x = 1$$

Thus, we see that since the left and right limits match, the limit of $f(x)$ at 1 exists, and is equal to 1. Furthermore, note that $f(1) = 1$. Thus, we see that

$$\lim_{x\to 1} f(x) = 1 = f(1)$$

which means that $f(x)$ is continuous at 1.

(4) Calculate the following derivatives using the limit definition of the derivative. You may NOT use L'Hospital's rule for these.

(a) [5 pts] $f'(x)$, where $f(x) = x^2 - 2$.

**Solution:**

By definition,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{((x+h)^2-2)-(x^2-2)}{h} \\ &= \lim_{h\to 0} \frac{x^2+2xh+h^2-2-x^2+2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} \\ &= \lim_{h\to 0}(2x+h) = \boxed{2x} \end{aligned}$$

(b) [5 pts] $f'(1)$, where $f(x) = \frac{1}{\sqrt{x}}$.

**Solution:**

By definition,

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{1+h}}-1}{h} \\ &= \lim_{h\to 0} \frac{\frac{1-\sqrt{1+h}}{\sqrt{1+h}}}{h} = \lim_{h\to 0} \frac{1-\sqrt{1+h}}{h\sqrt{1+h}} \end{aligned}$$

Using our standard difference of squares trick,

$$\begin{aligned} \lim_{h\to 0} \frac{1-\sqrt{1+h}}{h\sqrt{1+h}} &= \lim_{h\to 0} \frac{1-\sqrt{1+h}}{h\sqrt{1+h}} \cdot \frac{1+\sqrt{1+h}}{1+\sqrt{1+h}} \\ &= \lim_{h\to 0} \frac{1-(1+h)}{h\sqrt{1+h}(1+\sqrt{1+h})} \\ &= \lim_{h\to 0} \frac{-h}{h\sqrt{1+h}(1+\sqrt{1+h})} = \lim_{h\to 0} \frac{-1}{\sqrt{1+h}(1+\sqrt{1+h})} \end{aligned}$$

At this point, we can just plug in, getting

$$f'(1) = \lim_{h\to 0} \frac{-1}{\sqrt{1}(1+\sqrt{1})} = \boxed{-\frac{1}{2}}$$

(5) Calculate the following derivatives using whichever tools you wish. State the results you're using. You do NOT need to simplify your answers!

(a) [5 pts] Find $f'(x)$, if $f(x) = \ln(x)e^x + \sin(x)$.

**Solution:**

Using the product rule and sum rule,

$$\begin{aligned} f'(x) &= (\ln(x)e^x + \sin(x))' = (\ln(x)e^x)' + (\sin(x))' \\ &= (\ln(x))'e^x + \ln(x)(e^x)' + \cos(x) \\ &= \boxed{\frac{1}{x}e^x + \ln(x)e^x + \cos(x)} \end{aligned}$$

(b) [5 pts] Find $f'(x)$, if $f(x) = \dfrac{\tan(e^x)}{x^2+1}$

**Solution:**

Using the quotient rule,

$$\begin{aligned} f'(x) &= \frac{(\tan(e^x))'(x^2+1) - \tan(e^x)(x^2+1)'}{(x^2+1)^2} \\ &= \frac{(\tan(e^x))'(x^2+1) - 2x\tan(e^x)}{(x^2+1)^2} \end{aligned}$$

Now, using the chain rule,

$$\tan(e^x)' = \sec^2(e^x)\cdot(e^x)' = \sec^2(e^x)e^x$$

Plugging in, this gets

$$f'(x) = \boxed{\frac{\sec^2(e^x)e^x(x^2+1) - 2x\tan(e^x)}{(x^2+1)^2}}$$

(c) [5 pts] Find $f'(x)$, if $f(x) = x^2\cos(x)^{\sin(x)+1}$

**Solution:**

This requires logarithmic differentiation. If $y = x^2\cos(x)^{\sin(x)+1}$, then

$$\begin{aligned} \ln(y) &= \ln\left(x^2\cos(x)^{\sin(x)+1}\right) = \ln(x^2) + \ln\left(\cos(x)^{\sin(x)+1}\right) \\ &= 2\ln(x) + (\sin(x)+1)\ln(\cos(x)) \end{aligned}$$

Therefore, taking derivatives of both sides (using implicit differentiation, the product rule, and then the chain rule),

$$\begin{aligned} \frac{y'}{y} &= \frac{2}{x} + (\sin(x)+1)\ln(\cos(x))' + (\sin(x)+1)'\ln(\cos(x)) \\ &= \frac{2}{x} + (\sin(x)+1)\frac{1}{\cos(x)}(-\sin(x)) + \cos(x)\ln(\cos(x)) \\ &= \frac{2}{x} - \frac{\sin(x)^2 + \sin(x)}{\cos(x)} + \cos(x)\ln(\cos(x)) \end{aligned}$$

Thus, solving for $y'$, and plugging in $y$:

$$f'(x) = y' = y\left(\frac{2}{x} - \frac{\sin(x)^2 + \sin(x)}{\cos(x)} + \cos(x)\ln(\cos(x))\right)$$
$$= \boxed{x^2\cos(x)^{\sin(x)+1}\left(\frac{2}{x} - \frac{\sin(x)^2 + \sin(x)}{\cos(x)} + \cos(x)\ln(\cos(x))\right)}$$

(d) [5 pts] Find $y'$ in terms of $x$ and $y$, if $xy + e^y = \arctan(x)$.

**Solution:**

Using implicit differentiation to take derivatives of both sides,

$$(xy + e^y)' = (\arctan(x))'$$
$$\Rightarrow xy' + y + y'e^y = \frac{1}{1+x^2}$$

Now, solving for $y'$, we get

$$\Rightarrow y'(x + e^y) = \frac{1}{1+x^2} - y$$
$$\Rightarrow y' = \boxed{\frac{\frac{1}{1+x^2} - y}{x + e^y}}$$

(e) [5 pts] Find $g'(x)$, if $g(x) = \arccos(x) \cdot \int_1^x e^{t^2}\sin(\cos(t))\,dt$

**Solution:**

Using the product rule,

$$g'(x) = (\arccos(x))'\int_1^x e^{t^2}\sin(\cos(t))\,dt + \arccos(x)\left(\int_1^x e^{t^2}\sin(\cos(t))\,dt\right)'$$
$$= -\frac{1}{\sqrt{1-x^2}}\int_1^x e^{t^2}\sin(\cos(t))\,dt + \arccos(x)\left(\int_1^x e^{t^2}\sin(\cos(t))\,dt\right)'$$

Furthermore, by the Fundamental Theorem of Calculus,

$$\left(\int_1^x e^{t^2}\sin(\cos(t))\,dt\right)' = e^{x^2}\sin(\cos(x))$$

Therefore,

$$g'(x) = \boxed{-\frac{1}{\sqrt{1-x^2}}\int_1^x e^{t^2}\sin(\cos(t))\,dt + \arccos(x)e^{x^2}\sin(\cos(x))}$$

(f) [5 pts] Find $g'(x)$, if $g(x) = \int_1^{x^2+1}(u^2+u)\,du$

**Solution:**

Using the chain rule,

$$g'(x) = \left(\int_1^{x^2+1}(u^2+u)\,du\right)' = ((x^2+1)^2 + (x^2+1))\cdot(x^2+1)'$$
$$= \boxed{2x((x^2+1)^2 + x^2 + 1)}$$

(6) Calculate the equations of the following tangent lines:

(a) The tangent line to $y = \dfrac{e^{x-1}}{\ln(x)+1}$ at $x = 1$.

**Solution:**

To find the slope of the tangent line, use the derivative. Using the quotient rule,

$$y' = \frac{e^{x-1}(\ln(x)+1) - e^{x-1}\frac{1}{x}}{(\ln(x)+1)^2}$$

Therefore, at $x = 1$ the slope is

$$y'(1) = \frac{e^0(\ln(1)+1) - e^0\frac{1}{1}}{(\ln(1)+1)^2} = \frac{1-1}{1^2} = 0$$

To find the point on the graph that the tangent line goes through, we find the point with $x$-coordinate 1. Since

$$y(1) = \frac{e^0}{\ln(1)+1} = \frac{1}{1} = 1$$

the point on the line is $(1, 1)$.
Using the point-slope formula, we see that the equation is

$$y - 1 = 0 \cdot (x - 1) = 0$$

so the equation is $y - 1 = 0$, or $\boxed{y = 1}$.

(b) The tangent line to $y = f(x)g(x)$ at $x = 0$, given that $f(0) = 2, g(0) = 3, f'(0) = -1$, and $g'(0) = 4$.

**Solution:**

Like above, we need the slope of the tangent and the point on the line. Using the product rule,

$$y' = (f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

so we have that

$$y'(0) = f'(0)g(0) + f(0)g'(0) = (-1) \cdot 3 + 2 \cdot 4 = 5$$

To find the point on the line, find the $y$-coordinate at $x = 0$:

$$y(0) = f(0)g(0) = 2 \cdot 3 = 6$$

Therefore, the point on the line is $(0, 6)$. Thus, using the point-slope formula, we get

$$(y - 6) = 5(x - 0)$$
$$\Rightarrow y - 6 = 5x$$

Therefore, the equation of the line is $\boxed{y = 5x + 6}$.

(7) (a) [5 pts] Find the linearization of $f(x) = x^{1/3}$ at $x = 27$.

**Solution:**

The formula for the linearization of $f(x)$ at $x = a$ is

$$L(x) = f(a) + f'(a)(x - a)$$

Here, $f(x) = x^{1/3}$ and $a = 27$. Therefore, $f'(x) = \frac{1}{3}x^{-2/3}$. Calculating,

$$f(a) = 27^{1/3} = \sqrt[3]{27} = 3$$

$$f'(a) = \frac{1}{3}(27)^{-2/3} = \frac{1}{3}\left(27^{1/3}\right)^{-2} = \frac{1}{3}\left(\sqrt[3]{27}\right)^{-2}$$

$$= \frac{1}{3}3^{-2} = \frac{1}{3} \cdot \frac{1}{3^2} = \frac{1}{27}$$

Plugging these values in, we get that

$$\boxed{L(x) = 3 + \frac{1}{27}(x - 27)}$$

(b) [5 pts] Use the result from part (a) to estimate $\sqrt[3]{29}$.

**Solution:**

Since 29 is near 27, $f(29) \approx L(29)$. Therefore,

$$\sqrt[3]{29} = f(29) \approx L(29) = 3 + \frac{1}{27}(29 - 27) = 3 + \frac{2}{27} = \boxed{\frac{83}{27}}$$

(c) [5 pts] Could you use the result from (a) to estimate $\sqrt[3]{65}$, or would you need to do something different? (If you need to do something different, please eplain what it is.)

**Solution:**

Since 65 is not near 27, $L(65)$ isn't going to be particularly close to $\sqrt[3]{65}$. (If you calculate both values on your calculator, you'll see that it's true.) Instead, what we would need here is a linearization at point that's closer to 65. Since $\sqrt[3]{64} = 4$, the best thing to do would be to find the linearization at 64 and use that to estimate $\sqrt[3]{65}$.

(8) [10 pts] A sphere is expanding, with its volume growing at a rate of $4\text{ft}^3/\text{sec}$. How quickly is its surface area changing, when the volume of the sphere is $36\pi$ $\text{ft}^3$?

You may use the following formulas for the surface area and volume of a sphere with radius $r$:

$$A = 4\pi r^2, V = \frac{4}{3}\pi r^3$$

**Solution:**

This is clearly a related rates problem. This means we're going to go through our standard algorithm. We don't really need a picture here – our picture will just be a sphere with radius $r$, volume $V$, and surface area $A$.

(a) *Given:* $V' = 4, V = 36\pi$.
*Find:* $A'$

(b) *Relationships:* As given above,

$$A = 4\pi r^2$$
$$V = \frac{4}{3}\pi r^3$$

(c) *Differentiate:* Differentiating both sides of the relationships using chain rule,

$$A' = 8\pi r r'$$
$$V' = 4\pi r^2 r'$$

(d) *Solve for $A'$, plug in instantaneous info:* Here, we already have that $A' = 8\pi r r'$. Thus, to calculate $A'$ we need $r$ and $r'$. This is where we use the instantaneous info. Since $V = \frac{4}{3}\pi r^3$, and at the instant $V = 36\pi$, we get that

$$36\pi = \frac{4}{3}\pi r^3$$
$$\Rightarrow r^3 = 36\pi \cdot \frac{3}{4\pi}$$
$$\Rightarrow r^3 = 25$$

Taking cube roots, we get that $r = 3$. Now, since $V' = 4$, we can plug in $r = 3$ into the expression for $V'$ to solve for $r'$. Thus,

$$4 = 4\pi r^2 r' = 36\pi r'$$
$$\Rightarrow r' = \frac{4}{36\pi} = \frac{1}{9\pi}$$

Since we now know that at the instant, $r = 3$ and $r' = \frac{1}{9\pi}$, we can plug that into the expression for $A'$. Therefore,

$$A' = 8\pi r r' = 8\pi \cdot 3 \cdot \frac{1}{9\pi} = \frac{24\pi}{9\pi} = \frac{8}{3}$$

Thus, the surface area is changing at a rate of $8/3$ $\text{ft}^2/\text{sec}$.

(9) [10 pts] Let $f(x) = x^3 + 6x^2 + 9x + 7$. Find the absolute minimum value and absolute maximum value of $f$ on the interval $[-4, 2]$.

**Solution:**

Since we're finding absolute extrema on a closed interval, we can use the closed interval test: we find all the critical points and plug in the critical points which lie in the interval as well as the endpoints into $f$. The smallest number we get will be the absolute minimum, and the largest number will be the absolute maximum.

A critical point is a point where $f'$ is either 0 or doesn't exist. Here, we have that

$$f'(x) = 3x^2 + 12x + 9$$

Clearly, $f'(x)$ exists everywhere, so the only critical points will occur where $f'(x) = 0$. Solving,

$$\begin{aligned} 0 &= f'(x) = 3x^2 + 12x + 9 = 3(x^2 + 4x + 3) \\ \Rightarrow 0 &= x^2 + 4x + 3 = (x+1)(x+3) \end{aligned}$$

Thus, the only critical points are $x = -1$ and $x = -3$ which are both in the interval $[-4, 2]$. Thus, we need to plug in these critical points as well as the endpoints $-4$ and 2 into $f$. We get

$$\begin{aligned} f(-4) &= (-4)^3 + 6(-4)^2 + 9(-4) + 7 = 3 \\ f(-3) &= (-3)^3 + 6(-3)^2 + 9(-3) + 7 = 7 \\ f(-1) &= (-1)^3 + 6(-1)^2 + 9(-1) + 7 = 3 \\ f(2) &= (2)^3 + 6(2)^2 + 9(2) + 7 = 57 \end{aligned}$$

Therefore, $\boxed{\text{the absolute minimum is } 3}$, attained at $-4$ and $-1$, and $\boxed{\text{the absolute maximum is } 57}$, attained at 2.

(10) Let $f(x) = \dfrac{e^x}{x-1}$. Answer the following questions about $f(x)$.

(a) [5 pts] Find all the critical points of $f(x)$.

**Solution:**

A critical point is a value of $x$ such that $f'(x) = 0$ or doesn't exist. Using the quotient rule,

$$f'(x) = \frac{e^x(x-1) - e^x}{(x-1)^2} = \frac{e^x(x-2)}{(x-1)^2}$$

To find where $f'(x) = 0$, set the numerator to 0. Then,

$$e^x(x-2) = 0 \Rightarrow e^x = 0 \text{ or } x - 2 = 0$$

Since $e^x = 0$ is impossible, we see that $f'(x) = 0$ only if $x = 2$. To find where $f'(x)$ doesn't exist, set the denominator to 0. Thus, we get

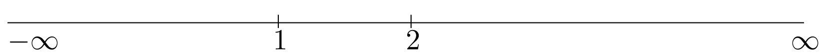
$$(x-1)^2 = 0 \Rightarrow x = 1$$

However, $x = 1$ is not in the domain of $f$, and therefore is not a critial point. Thus, the only critical point is $\boxed{x = 2}$ (although we will also need to use $x = 1$ for the next parts of the question.)

(b) [5 pts] Find the intervals on which $f(x)$ is increasing and decreasing.

**Solution:**

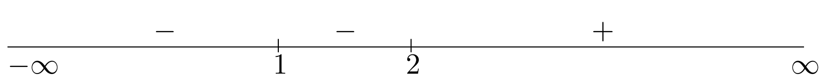
We make a number line chart for $f'$:




To fill in the signs of $f'$, plug in points in each interval and check the sign. We'll plug in 0 for the interval $(-\infty, 1)$, 1.5 for $(1, 2)$, and 3 for $(2, \infty)$. Calculating,

$$f'(0) = \frac{e^0(0-2)}{(0-1)^2} = -2 < 0$$
$$f'(1.5) = \frac{e^{1.5}(1.5-2)}{(1.5-1)^2} < 0$$
$$f'(3) = \frac{e^3(3-2)}{(3-1)^2} > 0$$

Thus, filling in the pluses and minuses on the number line, we get




Therefore, $f(x)$ is $\boxed{\text{decreasing on } (-\infty, 1) \cup (1, 2) \text{ and increasing on } (2, \infty)}$.

(c) [5 pts] Find the intervals on which $f(x)$ is concave up and concave down.

**Solution:**

Here, we need to figure out where $f''(x)$ is positive and negative. Calculating,

$$f''(x) = \left(\frac{e^x(x-2)}{(x-1)^2}\right)' = \frac{(e^x(x-2))'(x-1)^2 - e^x(x-2)2(x-1)}{(x-1)^4}$$
$$= \frac{(e^x(x-2))'(x-1) - 2e^x(x-2)}{(x-1)^3}$$

Continuing to simplify,

$$f''(x) = \frac{(e^x(x-2) + e^x)(x-1) - 2e^x(x-2)}{(x-1)^3}$$
$$= \frac{e^x(x-1)(x-1) - 2e^x(x-2)}{(x-1)^3}$$
$$= \frac{e^x(x^2 - 2x + 1 - 2x + 4)}{(x-1)^3}$$
$$= \frac{e^x(x^2 - 4x + 5)}{(x-1)^3}$$

We need to find where $f''(x) = 0$ or $f''(x)$ doesn't exist. To get $f''(x) = 0$, set the numerator to 0.

$$e^x(x^2 - 4x + 5) = 0 \Rightarrow e^x = 0 \text{ or } x^2 - 4x + 5 = 0$$

Since $e^x = 0$ is impossible, and $x^2 - 4x + 5 = (x-2)^2 + 1$ is always positive, we see there are no solutions to $f''(x) = 0$.
To find where $f''(x)$ doesn't exist, set the denominator to 0. Then,

$$(x-1)^3 = 0 \Rightarrow x = 1$$

Thus, the only point on our sign chart is $x = 1$. Starting the sign chart:

$-\infty$ ——————— $1$ ——————— $\infty$

We plug in 0 to test the interval $(-\infty, 1)$ and plug in 2 to test the interval $(1, \infty)$:

$$f''(0) = \frac{e^0(0^2 - 4\cdot 0 + 5)}{(0-1)^3} = \frac{5}{-1} < 0$$
$$f''(2) = \frac{e^2(2^2 - 4\cdot 2 + 5)}{(2-1)^3} = \frac{e^2}{1} > 0$$

Thus, the sign chart is

$-\infty$ ——— $-$ ——— $1$ ——— $+$ ——— $\infty$

Finally, the conclusion is that $f(x)$ is **concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.**

(d) [5 pts] Find the horizontal asymptotes of $f(x)$. For each asymptote, state whether it occurs at $\infty$ or $-\infty$.

**Solution:**

To find horizontal asymptotes, we take the limit as $x$ approaches $\infty$ and $-\infty$ of $f(x)$. Taking the limit as $x \to \infty$ first, we see that $\lim_{x\to\infty} \frac{e^x}{x-1}$ is of the form $\frac{\infty}{\infty}$, so we can use L'Hospital's. Hence,

$$\lim_{x\to\infty} \frac{e^x}{x-1} = \lim_{x\to\infty} \frac{e^x}{1} = \lim_{x\to\infty} e^x = \infty$$

Since the answer is not a number, there's no asymptote at $\infty$.

Now, consider $\lim_{x\to-\infty}\frac{e^x}{x-1}$. As $x\to-\infty$, $e^x$ approaches 0, and $x-1$ approaches $-\infty$. Thus, this is of the form $\frac{0}{-\infty}$ which is clearly 0. Therefore,

$$\lim_{x\to-\infty}\frac{e^x}{x-1}=0$$

Thus, there is an asymptote at $-\infty$, and it is $\boxed{y=0}$.

(e) [5 pts] Find the vertical asymptotes of $f(x)$. For each vertical asymptotes $x=a$, calculate $\lim_{x\to a^+}f(x)$ and $\lim_{x\to a^-}f(x)$.

**Solution:**

A vertical asymptote $a$ is a place where

$$\lim_{x\to a^-}f(x)=\pm\infty \text{ or } \lim_{x\to a^+}f(x)=\pm\infty$$

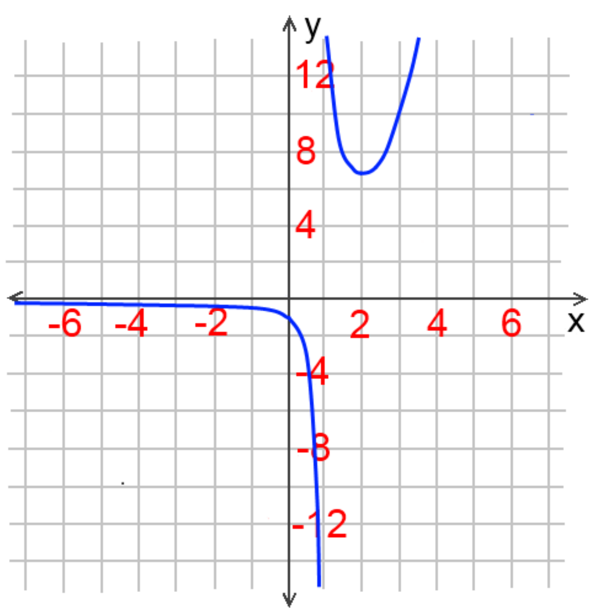
Here, they occur where the denominator of $f(x)$ is equal to 0. Setting $x-1$ to 0, we see that the only possibility is $\boxed{x=1}$. When $x=1$, the numerator is $e^1\neq 0$, so $x=1$ is indeed an asymptote.
Now, doing the calculation of the limits:

$$\lim_{x\to1^-}f(x)=\lim_{x\to1^-}\frac{e^x}{x-1}=\frac{e}{0^-}=-\infty$$
$$\lim_{x\to1^+}f(x)=\lim_{x\to1^+}\frac{e^x}{x-1}=\frac{e}{0^+}=\infty$$

(f) [5 pts] Use the information from the previous parts of the question to sketch the graph of $f(x)$.

**Solution:**

Putting together all the information, and also graphing a couple of points, yields the following sketch:



(11) [10 pts] Find the point on the parabola $y = x^2 - 2$ that is closest to the origin (that is, to the point $(0, 0)$).

**Solution:**

Let the point on the parabola be $(x, y)$. We want to minimize the distance to the origin, which is

$$d = \sqrt{x^2 + y^2}$$

Since distance is positive, let's instead minimize $f = d^2 = x^2 + y^2$.

We need to find relationships between the variables enabling us to write $f$ in terms of one variable. Here, since $(x, y)$ is on the parabola $y = x^2 - 2$, we get that relationship, and therefore,

$$f = x^2 + (x^2 - 2)^2$$

Since the $x$ coordinate can be anything, the domain of $f(x)$ is $(-\infty, \infty)$.

To find the minimum value of $f(x)$, let us first find the critical points of $f$. Since $f(x) = x^2 + (x^2 - 2)^2 = x^2 + x^4 - 4x^2 + 4 = x^4 - 3x^2 + 4$, we have that

$$f'(x) = 4x^3 - 6x = 2x(2x^2 - 3)$$

$f'(x)$ clearly exists everywhere. Thus, the only critical points occur where $f'(x) = 0$. Solving,

$$0 = f'(x) = 2x(2x^2 - 3) \Rightarrow x = 0 \text{ or } 2x^2 = 3$$

Thus, the critical points are $x = 0$ and $x = \pm\sqrt{\frac{3}{2}}$.

Let us now use the first derivative to find the absolute minimum of $f$. Making the sign chart for $f'$, we get:

| $-\infty$ | | $-\sqrt{\frac{3}{2}}$ | | $0$ | | $\sqrt{\frac{3}{2}}$ | | $\infty$ |
|---|---|---|---|---|---|---|---|---|
| | $-$ | | $+$ | | $-$ | | $+$ | |

Thus, $f$ is decreasing on $(-\infty, -\sqrt{3/2})$, increasing on $(-\sqrt{3/2}, 0)$, decreasing on $(0, \sqrt{3/2})$, and increasing on $(\sqrt{3/2}, \infty)$. Sketching this, this makes it clear that the only places an absolute minimum might be attained are at $\pm\sqrt{3/2}$.

Checking, we see that

$$f(\sqrt{3/2}) = \sqrt{3/2}^2 + (\sqrt{3/2}^2 - 2)^2 = \frac{3}{2} + \left(\frac{3}{2} - 2\right)^2 = \frac{3}{2} + \frac{1}{4} = \frac{7}{4}$$

$$f(-\sqrt{3/2}) = (-\sqrt{3/2})^2 + ((-\sqrt{3/2})^2 - 2)^2 = \frac{3}{2} + \left(\frac{3}{2} - 2\right)^2 = \frac{3}{2} + \frac{1}{4} = \frac{7}{4}$$

Thus, we see that an absolute minimum is attained both at $\sqrt{3/2}$ and at $-\sqrt{3/2}$. Since we're asked for both coordinates, we need to calculate the $y$-value at those points. Calculating,

$$y(\sqrt{3/2}) = (\sqrt{3/2})^2 - 2 = \frac{3}{2} - 2 = -\frac{1}{2}$$

$$y(-\sqrt{3/2}) = (-\sqrt{3/2})^2 - 2 = \frac{3}{2} - 2 = -\frac{1}{2}$$

Therefore, the two points at which the distance from the origin is minimal are

$$\boxed{(-\sqrt{3/2}, -1/2) \text{ and } (\sqrt{3/2}, -1/2).}$$

(12) Solve the following problems:

(a) [5 pts] Find the general expression for a function $F(x)$ such that $F'(x) = e^{2x} - \sin(x) + \frac{1}{1+x^2}$.

**Solution:**

Using the rules for antiderivatives, we see that the general expression is

$$F(x) = \boxed{\frac{e^{2x}}{2} + \cos(x) + \arctan(x) + C}$$

You can check that this works by taking the derivative of $F(x)$ and making sure you get the right answer.

(b) [5 pts] Find the function $F(x)$ such that $F'(x) = 2x + 1$ and $F(1) = 3$.

**Solution:**

The general antiderivative of $2x + 1$ is

$$F(x) = x^2 + x + C$$

To solve for $C$, we use the fact that $F(1) = 3$. Plugging in,

$$3 = F(1) = 1^2 + 1 + C = C + 2$$
$$\Rightarrow C = 1$$

Therefore,

$$F(x) = \boxed{x^2 + x + 1}$$

(c) Find the function $F(x)$ such that $F''(x) = 1 + \frac{1}{x^2}$, with $F'(1) = 1$ and $F(1) = 2$.

**Solution:**

Here, we need to take an antiderivative twice. $F'(x)$ will be an antiderivative of $F''(x) = 1 + \frac{1}{x^2}$, and $F(x)$ is the antiderivative of $F'(x)$. Using the general expression for an antiderivative of $1 + \frac{1}{x^2}$, we see that

$$F'(x) = x - \frac{1}{x} + C$$

and taking the general antiderivative of this, we get

$$F(x) = \frac{x^2}{2} - \ln(x) + Cx + D$$

Now, we need to use the conditions to solve for $C$ and $D$. Since $1 = F'(1)$,

$$1 = F'(1) = 1 - \frac{1}{1} + C$$
$$\Rightarrow C = 1$$

Now, using the fact that $F(1) = 2$ we can solve for $D$:

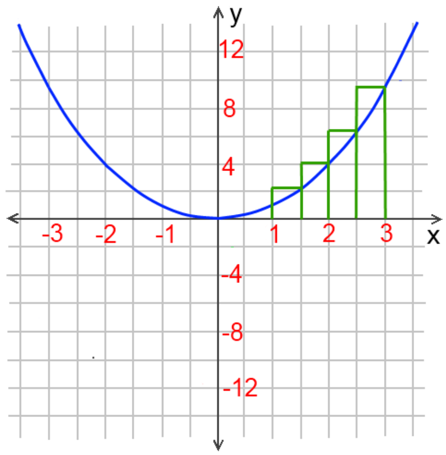
$$2 = F(1) = \frac{1}{2} - \ln(1) + 1 + D = D + \frac{3}{2}$$
$$\Rightarrow \frac{1}{2} = D$$

Therefore, $\boxed{F(x) = \frac{x^2}{2} - \ln(x) + x + \frac{1}{2}.}$

(13) Solve the following problems:

(a) [5 pts] Estimate the area under $y = x^2$ from $x = 1$ to $x = 3$ using 4 rectangles and the right endpoint rule. Use the graph to explain whether this an underestimate or an overestimate.

**Solution:**




Here, we have $\Delta x = \frac{b-a}{n} = \frac{2}{4} = \frac{1}{2}$. Therefore,

$$x_0 = 1, x_1 = \frac{3}{2}, x_2 = 2, x_3 = \frac{5}{2}, x_4 = 3$$

Thus, since we're using right endpoints.

$$\text{height of first rectangle} = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

$$\text{height of second rectangle} = 2^2 = 4$$

$$\text{height of third rectangle} = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$$

$$\text{height of fourth rectangle} = 3^2 = 9$$

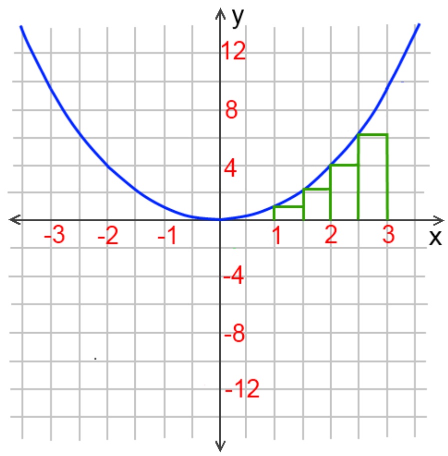
Since the base of each rectangle is $\frac{1}{2}$, the total area is

$$\frac{1}{2} \cdot \frac{9}{4} + \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot \frac{25}{4} + \frac{1}{2} \cdot 9 = \frac{9 + 16 + 25 + 36}{8} = \frac{86}{8} = \boxed{\frac{43}{4}}$$

From the picture, we see that the area of the rectangles contains the area under the curve. Therefore, this is an $\boxed{\text{overestimate.}}$

(b) [5 pts] Estimate the area under $y = x^2$ from $x = 1$ to $x = 3$ using 4 rectangles and the left endpoint rule. Use the graph to explain whether this an underestimate or an overestimate.

**Solution:**




Like in part (a), we have $\Delta x = \frac{b-a}{n} = \frac{2}{4} = \frac{1}{2}$ and

$$x_0 = 1, x_1 = \frac{3}{2}, x_2 = 2, x_3 = \frac{5}{2}, x_4 = 3$$

Thus, since we're using left endpoints.

$$\text{height of first rectangle} = 1^2 = 1$$

$$\text{height of second rectangle} = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

$$\text{height of third rectangle} = 2^2 = 4$$

$$\text{height of fourth rectangle} = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$$

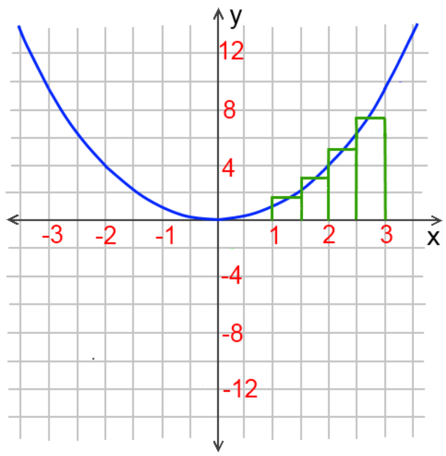
Since the base of each rectangle is $\frac{1}{2}$, the total area is

$$\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{9}{4} + \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot \frac{25}{4} = \frac{4 + 9 + 16 + 25}{8} = \frac{54}{8} = \boxed{\frac{27}{4}}$$

From the picture, we see that the area of the rectangles is contained in the area under the curve. Therefore, this is an $\boxed{\text{underestimate.}}$

(c) [5 pts] Estimate the area under $y = x^2$ from $x = 1$ to $x = 3$ using 4 rectangles and the midpoint rule. Is it immediately clear whether this is an underestimate or an overestimate?

**Solution:**




Like in part (a), we have $\Delta x = \frac{b-a}{n} = \frac{2}{4} = \frac{1}{2}$ and

$$x_0 = 1, x_1 = \frac{3}{2}, x_2 = 2, x_3 = \frac{5}{2}, x_4 = 3$$

Since the sample points are midpoints, we have that

$$x_1^* = \frac{x_0 + x_1}{2} = \frac{5}{4}, x_2^* = \frac{x_1 + x_2}{2} = \frac{7}{4}$$
$$x_3^* = \frac{x_2 + x_3}{2} = \frac{9}{4}, x_4^* = \frac{x_3 + x_4}{2} = \frac{11}{4}$$

Therefore, plugging these in,

$$\text{height of first rectangle} = (x_1^*)^2 = \left(\frac{5}{4}\right)^2 = \frac{25}{16}$$

$$\text{height of second rectangle} = (x_2^*)^2 = \left(\frac{7}{4}\right)^2 = \frac{49}{16}$$

$$\text{height of third rectangle} = (x_3^*)^2 = \left(\frac{9}{4}\right)^2 = \frac{81}{16}$$

$$\text{height of fourth rectangle} = (x_4^*)^2 = \left(\frac{11}{4}\right)^2 = \frac{121}{16}$$

Since the base of each rectangle is $\frac{1}{2}$, the total area is

$$\frac{1}{2} \cdot \frac{25}{16} + \frac{1}{2} \cdot \frac{49}{16} + \frac{1}{2} \cdot \frac{81}{16} + \frac{1}{2} \cdot \frac{121}{16} = \frac{25 + 49 + 81 + 121}{32} = \frac{276}{32} = \boxed{\frac{69}{8}}$$

From the picture, we see that it's not clear whether it's an overestimate or an underestimate.

(14) Solve the following problems:

(a) [5 pts] Express the sum $\frac{1}{4}+\frac{1}{5}+\cdot+\frac{1}{10}$ using sigma notation.

**Solution:**

This sum is clearly

$$\boxed{\sum_{i=4}^{10}\frac{1}{i}}$$

(b) [10 pts] Use the limit of Riemann sums with right endpoints to calculate the integral $\int_0^2 (x^2+1)\,dx$. You may use the formula

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

**Solution:**

Here, we need to calculate an expression for the $n$th Riemann sum with right endpoints, then take the limit as $n \to \infty$.
For the $n$th Riemann sum, we have $\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$. Since $x_0 = a = 0$, we therefore have

$$x_0 = 0, x_1 = \frac{2}{n}, x_2 = \frac{4}{n}, \ldots, x_n = \frac{2n}{n} = 2$$

Therefore, since we're using right endpoints,

$$\text{height of first rectangle} = f(x_1) = 1 + x_1^2 = 1 + \left(\frac{2}{n}\right)^2$$

$$\text{height of second rectangle} = f(x_1) = 1 + x_2^2 = 1 + \left(\frac{4}{n}\right)^2$$

$$\vdots$$

$$\text{height of } n\text{th rectangle} = f(x_1) = 1 + x_n^2 = 1 + \left(\frac{2n}{n}\right)^2$$

Thus, the whole Riemann sum is

$$\frac{2}{n}\left(1+\left(\frac{2}{n}\right)^2\right) + \frac{2}{n}\left(1+\left(\frac{4}{n}\right)^2\right) + \cdots + \frac{2}{n}\left(1+\left(\frac{2n}{n}\right)^2\right)$$

and writing it in sigma notation and simplifying,

$$\begin{aligned}\sum_{i=1}^{n}\frac{2}{n}\left(1+\left(\frac{2i}{n}\right)^2\right) &= \sum_{i=1}^{n}\frac{2}{n}\left(1+\frac{4i^2}{n^2}\right) = \sum_{i=1}^{n}\left(\frac{2}{n}+\frac{8i^2}{n^3}\right) \\ &= \sum_{i=1}^{n}\frac{2}{n} + \sum_{i=1}^{n}\frac{8i^2}{n^3} = \frac{2}{n}\sum_{i=1}^{n}1 + \frac{8}{n^3}\sum_{i=1}^{n}i^2 \\ &= \frac{2}{n}\cdot n + \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\end{aligned}$$

Thus, continuing to simplify, the $n$th Riemann sum $R_n$ is

$$R_n = 2 + \frac{4(n+1)(2n+1)}{3n^2} = 2 + \frac{4(2n^2+3n+1)}{3n^2}$$
$$= 2 + \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}$$

Finally, we need to take the limit of $R_n$ as $n \to \infty$. Clearly, $\frac{4}{n}$ and $\frac{4}{3n^2}$ approach 0 as $n$ approaches $\infty$, so

$$\lim_{n\to\infty} R_n = 2 + \frac{8}{3} = \frac{14}{3}$$

Thus, we have shown using Riemann sums that

$$\boxed{\int_0^2 (x^2+1)\,dx = \frac{14}{3}}$$

(15) Find the values of the following definite integrals, using whichever tools you choose. State the results you're using.

(a) [5 pts]

$$\int_{-1}^{2} e^x - x\,dx$$

**Solution:**

By the Fundamental Theorem of Calculus, to evaluate $\int_a^b f(x)\,dx$ we need to find an antiderivative $F(x)$ of $f(x)$, and then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Thus, we need an an antiderivative of $e^x - x$. This is clearly $e^x - \frac{x^2}{2}$. Therefore,

$$\int_{-1}^{2} e^x - x\,dx = \left(e^x - \frac{x^2}{2}\right)\Big|_{-1}^{2} = e^2 - 2 - \left(e^{-1} - \frac{1}{2}\right)$$
$$= \boxed{e^2 - e^{-1} - \frac{3}{2}}$$

(b) [5 pts]

$$\int_{\pi/6}^{\pi} \cos(x)\,dx$$

**Solution:**

The antiderivative of $\cos(x)$ is $\sin(x)$. Thus, using the Fundamental Theorem of Calculus,

$$\int_{\pi/6}^{\pi} \cos(x)\,dx = \sin(x)\Big|_{\pi/6}^{\pi} = \sin(\pi) - \sin(\pi/6) = 0 - \frac{1}{2} = \boxed{-\frac{1}{2}}$$

(c) [5 pts]

$$\int_{1}^{e} 1 + \frac{1}{x}\,dx$$

**Solution:**

The antiderivative of $1 + \frac{1}{x}$ is $x + \ln(x)$. Thus,

$$\int_{1}^{e} 1 + \frac{1}{x}\,dx = (x + \ln(x))\Big|_{1}^{e} = e + \ln(e) - 1 - \ln(1)$$
$$= e + 1 - 1 - 0 = \boxed{e}$$