In-Class Questions for April 11th

Part 1:

1. Let $f(x) = \dfrac{x}{x^2 - 1}$.

   (a) Calculate $f'(x)$ and $f''(x)$.

   (b) Find the intervals on which $f(x)$ is increasing/decreasing and its local minimums/maximums.

   (c) Find the intervals on which $f(x)$ is concave up/down and its inflection points.

Proudly written by Corey. I can brag about this, right?

Part 1:

1. Let $f(x) = \frac{x}{x^2-1}$.

   (a) Calcu1ate $f'(x)$ and $f''(x)$.

$$
\begin{aligned}
f'(x) &= \frac{(x^2-2)(x)' - (x)(x^2-1)'}{(x^2-1)^2} \\
&= \frac{(x^2-1) - (2x^2)}{(x^2-1)^2} \\
&= -\frac{(x^2+1)}{(x^2-1)^2} \\
&= -\frac{(x^2+1)}{(x+1)(x-1)(x+1)(x-1)} \\
&= \boxed{-\frac{(x^2+1)}{(x+1)^2(x-1)^2}}
\end{aligned}
$$

$$
\begin{aligned}
f''(x) &= -\frac{(x^2-1)^2(x^2+1)' - (x^2+1)((x^2-1)^2)'}{(x^2-1)^4} \\
&= -\frac{(x^2-1)^2(2x) - (x^2+1)(2(x^2-1)(2x))}{(x^2-1)^4} \\
&= -\frac{(x^2-1)^2(2x) - (4x)(x^2+1)(x^2-1)}{(x^2-1)^4} \\
&= -\frac{(2x)(x^2-1)((x^2-1) - 2(x^2+1))}{(x^2-1)^4} \\
&= -\frac{(2x)(x^2-1-2x^2-2)}{(x^2-1)^3} \\
&= -\frac{(2x)(-x^2-3)}{(x^2-1)^3} \\
&= \boxed{\frac{(2x)(x^2+3)}{(x^2-1)^3}}
\end{aligned}
$$

(b) Find the intervals on which $f(x)$ is increasing/decreasing.

$$f'(x) = -\frac{(x^2+1)}{(x+1)^2(x-1)^2}$$

From the numerator we have:

$$\begin{aligned} 0 &= (x^2+1) \\ -1 &= x^2 \\ \sqrt{-1} &= x \end{aligned}$$

No real numbers make the numerator equal zero, so we disregard this value for x.

From the denominator we have:

$$\begin{aligned} 0 &= (x+1)^2(x-1)^2 \\ x &= 1 \text{ or } -1. \end{aligned}$$

To check if these are critical points, we see if they're in the domain of f.

As it turns out, they're not in the domain, so they're not critical points. This means that the graph of f may not go from increasing to decreasing, or vise versa, on either side of these points. Now we must check the intervals $(\infty, -1)$, $(-1, 1)$, and $(1, \infty)$.

Well, here it goes. So, $f'(-2) = -\frac{5}{9}$, $f'(0) = -1$, and $f'(2) = -\frac{5}{9}$. Hence, $f(x)$ is decreasing in each interval.

Bear in mind, that doesn't mean that the function is continuous, and, in this case, it clearly is not.

(c) Find the intervals on which $f(x)$ is concave up/down.

$$f(x) = \frac{(2x)(x^2+3)}{(x^2-1)^3}$$

We set the numerator and denominator equal to zero to find candidates for inflection points, which will partition our graph into intervals.

From the numerator, we find $x = 0$ will make $f''(x) = 0$.

From the denominator, we find that $x = -1$ or 1 will make $f''(x)$ be undefined.

Now we check $x = -1$, $x = 0$, and $x = 1$ to see if they are in the domain of our function.

Looking back at f:

$$f(x) = \frac{x}{x^2 - 1}$$

We see that 0 is in the domain of our function, but neither -1 nor 1 are.

From this, we can conclude that $(0, f(0))$, or $(0, 0)$, is an inflection point. However, just because the other two candidates were not in the domain doesn't mean that we don't use them to partition our intervals.
The intervals we check are $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, $(1, \infty)$.

$f''(-2)$ is negative.
$f''(-.5)$ is positive.
$f''(.5)$ is negative.
$f''(2)$ is positive.

That is to say, $f(x)$ is concave down for $(-\infty, -1)$ and $(0, 1)$, and $f(x)$ is concave up for $(-1, 0)$ and $(1, \infty)$.