In-Class Work Solutions for April 13th

Part 1:

1. Let $f(x) = \frac{x}{x^2-1}$.

   (a) Find the horizontal asymptotes of $f(x)$.

   Solution:

   By dividing both the numerator and the denominator by the highest degree of x in the denominator, we find that

   $$\lim_{x\to\infty} \frac{\frac{x}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1 - \frac{1}{x^2}} = \frac{\lim_{x\to\infty} \frac{1}{x}}{\lim_{x\to\infty} \left(1 - \frac{1}{x^2}\right)}$$
   $$= \frac{0}{1-0} = 0$$

   So, there is a horizontal asymptote $y = 0$ at ∞. Similarly,

   $$\lim_{x\to-\infty} \frac{\frac{x}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}} = \lim_{x\to-\infty} \frac{\frac{1}{x}}{1 - \frac{1}{x^2}} = \frac{\lim_{x\to-\infty} \frac{1}{x}}{\lim_{x\to-\infty} \left(1 - \frac{1}{x^2}\right)}$$
   $$= \frac{0}{1-0} = 0$$

   Thus, there's a horizontal asymptote $y = 0$ at $-\infty$.

   (b) Find the vertical asymptotes of $f(x)$. For each vertical asymptote $x = L$, find

   $$\lim_{x\to L^-} f(x) \text{ and } \lim_{x\to L^+} f(x)$$

   Solution:

   We begin by discovering what values of x make the denominator zero, we find where the function is undefined, and thus we find its vertical asymptotes.

   $x = 1$ or -1 make $f(x)$ undefined.

   Checking, the numerator isn't 0 at either of these, so they are definitely asymptotes. Now, working out the limits:

$$\lim_{x\to -1^-} \frac{x}{x^2-1} \approx \frac{-1.01}{(-1.01)^2-1} \approx \frac{-1}{\text{tiny positive number}} = \text{HUGE negative number}$$
$$\lim_{x\to -1^+} \frac{x}{x^2-1} \approx \frac{-0.99}{(-0.99)^2-1} \approx \frac{-1}{\text{tiny negative number}} = \text{HUGE positive number}$$
$$\lim_{x\to 1^-} \frac{x}{x^2-1} \approx \frac{0.99}{(0.99)^2-1} \approx \frac{1}{\text{tiny negative number}} = \text{HUGE negative number}$$
$$\lim_{x\to 1^+} \frac{x}{x^2-1} \approx \frac{1.01}{(1.01)^2-1} \approx \frac{1}{\text{tiny positive number}} = \text{HUGE positive number}$$

Therefore, combining all these,

$$\lim_{x\to -1^-} \frac{x}{x^2-1} = -\infty, \lim_{x\to -1^+} \frac{x}{x^2-1} = \infty$$
$$\lim_{x\to 1^-} \frac{x}{x^2-1} = -\infty \lim_{x\to 1^+} \frac{x}{x^2-1} = \infty$$

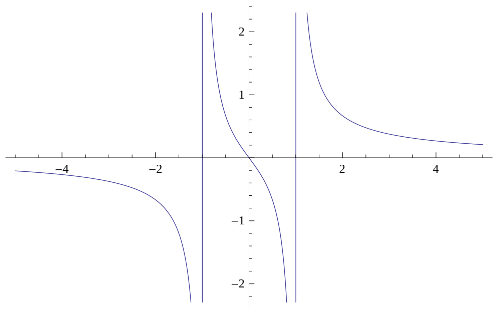
Part 2:

1. Let $f(x)$ be the same as in Part 1 above.

   (a) Use all the information you've gathered so far to make a sketch of $f(x)$:

   Solution:

   Here it is:



   (b) Sketch a function $g(x)$ that satisfies the following:

   - Concave up on $(-\infty, -2) \cup (2, \infty)$ and concave down on $(-2, 0) \cup (0, 2)$.
   - Horizontal asymptote $y = 0$ at ∞ and $y = 1$ at $-\infty$.
   - Local maxes at $(1, 1)$ and $(-1, 2)$.
   - A local min at $(0, 0)$.

   Solution:

   Here it is: