In-Class Work Solutions for April 6th

Part 1:

1. Let $f(x) = x^{2/3} + \dfrac{2x}{3}$.

   (a) Calculate $f'(x)$ and $f''(x)$.

   Solution:

   Calculating,

$$f'(x) = \left(x^{2/3} + \frac{2x}{3}\right)' = \left(x^{2/3} + \frac{2}{3}x\right)'$$
$$= \boxed{\frac{2}{3}x^{-1/3} + \frac{2}{3}}$$

   Differentiating again,

$$f''(x) = \left(\frac{2}{3}x^{-1/3} + \frac{2}{3}\right)' = \frac{2}{3} \cdot \left(-\frac{1}{3}\right) x^{-4/3}$$
$$= -\boxed{\frac{2}{9}x^{-4/3}}$$

   (b) Find the intervals on which $f(x)$ is increasing/decreasing.

   Solution:

   To do this, we find the places where $f'(x)$ could change sign, plot them on the number line, and find the sign of $f'(x)$ on each interval. $f'(x)$ changes signs only at places where $f'(x)$ doesn't exist or is 0. Simplifying a little,

$$f'(x) = \frac{2}{3}x^{-1/3} + \frac{2}{3} = \frac{2}{3\sqrt[3]{x}} + \frac{2}{3}$$
$$= \frac{2}{3\sqrt[3]{x}} + \frac{2}{3}\frac{\sqrt[3]{x}}{\sqrt[3]{x}}$$
$$= \frac{2 + 2\sqrt[3]{x}}{3\sqrt[3]{x}}$$

   <u>$f'(x)$ doesn't exist:</u> This happens if the denominator is 0. Here, this happens if

$$\sqrt[3]{x} = 0$$
$$\Rightarrow x = 0$$

$\underline{f'(x) = 0}$: This happens if the numerator is 0. Thus,

$$\begin{aligned} 2 + 2\sqrt[3]{x} &= 0 \\ \Rightarrow 2\sqrt[3]{x} &= -2 \\ \Rightarrow \sqrt[3]{x} &= -1 \\ \Rightarrow x &= (-1)^3 = -1 \end{aligned}$$

Therefore, we have two places where f' could change sign: $x = 0$ and $x = -1$. We need to test each of the intervals $(\infty, -1), (-1, 0)$ and $(0, \infty)$ to see the sign of f' on each one. We test -8 in $(-\infty, -1)$, $-1/8$ in $(-1, 0)$ and 1 in $(1, \infty)$ (these numbers were chosen to make cube roots easier!):

$$\begin{aligned} f'(-8) &= \frac{2 + 2\sqrt[3]{-8}}{3\sqrt[3]{-8}} = \frac{2 + 2 \cdot (-2)}{3 \cdot (-2)} \\ &= \frac{-2}{-6} = \frac{1}{3} > 0 \\ f'(-1/8) &= \frac{2 + 2\sqrt[3]{-1/8}}{3\sqrt[3]{-1/8}} = \frac{2 + 2 \cdot (-1/2)}{3 \cdot (-1/2)} \\ &= \frac{1}{-3/2} = -\frac{2}{3} < 0 \\ f'(1) &= \frac{2 + 2\sqrt[3]{1}}{3\sqrt[3]{1}} = \frac{2 + 2 \cdot 1}{3 \cdot 1} \\ &= \frac{4}{3} > 0 \end{aligned}$$

Therefore, we see that $f'(x)$ is positive on $(-\infty, -1)$ and $(0, \infty)$, and negative on $(-1, 0)$. Thus, $f(x)$ is $\boxed{\text{increasing on } (-\infty, 0) \text{ and } (0, \infty)}$ and $\boxed{\text{decreasing on } (-1, 0)}$.

(c) Find the intervals on which $f(x)$ is concave up/down.

Solution:

This question is just like part (a), except we use the the second derivative. Start by finding the places where $f''(x)$ could change sign, which are the places where $f''(x)$ doesn't exist or is equal to 0. Again, simplifying a little,

$$f''(x) = \frac{2}{9}x^{-4/3} = -\frac{2}{9x^{4/3}} = -\frac{2}{9(\sqrt[3]{x})^4}$$

$\underline{f'(x) \text{ doesn't exist}}$: This happens when the denominator is 0. It's easy to see that this is only possible if $x = 0$.

$\underline{f'(x) = 0}$: This happens when the numerator is 0. Since the numerator is never 0, this never happens.

Thus, there's only one places where $f''(x)$ could change sign, and that's at $x = 0$. We test the two intervals $(-\infty, 0)$ and $(0, \infty)$:

$$\begin{aligned} f''(-1) &= -\frac{2}{9(\sqrt[3]{-1})^4} = -\frac{2}{9 \cdot (-1)^4} \\ &= -\frac{2}{9} < 0 \\ f''(1) &= -\frac{2}{9(\sqrt[3]{1})^4} = -\frac{2}{9 \cdot (1)^4} \\ &= -\frac{2}{9} > 0 \end{aligned}$$

Thus, $f''(x)$ is negative everywhere, and therefore $f(x)$ is concave down everywhere. Since $f'(x)$ isn't defined at 0, it's probably best to write this as $\boxed{f(x) \text{ is concave down on } (-\infty, 0) \text{ and } (0, \infty)}$ although I won't be too picky about that!

2. **Second Derivative Test:** Fill in the blanks: for any function $f(x)$, if c satisfies $f'(c) = 0$, the following holds:

   - If $f''(c) > 0$, then $f(x)$ has a <u>local minimum</u> at c.
   - If $f''(c) < 0$, then $f(x)$ has a <u>local maximum</u> at c.

   If you're not sure, sketch a picture of $f(x)$ to see what's going on!