GROUP WORK
7 FEB. 2012

① Calculate the following limits using limit laws:

a) $\lim_{x \to -\infty} \frac{2x^3 + 3x^2 - x}{3x^3 - 3x + 4}$ (limit of a rational function)
- A rational function is any function that can be written as a ratio of two polynomial functions.

In our case,

$$f(x) = \frac{P(x)}{Q(x)} = \frac{2x^3 + 3x^2 - x}{3x^3 - 3x + 4}$$

and the degree of $q(x)$ is 3, which is equal to the degree of $Q(x)$, so we divide $P(x)$ and $Q(x)$ by x^n, where n is the highest degree of the denominator.

$$f(x) = \frac{2x^3 + 3x^2 - x}{3x^3 - 3x + 4} = \frac{\frac{2x^3}{x^3} + \frac{3x^2}{x^3} - \frac{x}{x^3}}{\frac{3x^3}{x^3} - \frac{3x}{x^3} + \frac{4}{x^3}} = \frac{2 + \frac{3}{x} - \frac{1}{x^2}}{3 - \frac{3}{x^2} + \frac{4}{x^3}}$$

and then take its limit:

$$\lim_{x \to -\infty} \frac{2 + \frac{3}{x} - \frac{1}{x}}{3 - \frac{3}{x^2} + \frac{4}{x^3}} = \frac{2 + 0 - 0}{3 - 0 + 0} = \boxed{\frac{2}{3}}.$$

b) $$\lim_{x \to \infty} \frac{x + 1}{2x^2 + x + 5} = \lim_{x \to \infty} \frac{\frac{x}{x^2} + \frac{1}{x^2}}{\frac{2x^2}{x^2} + \frac{x}{x^2} + \frac{5}{x^2}} = \lim_{x \to \infty} \frac{\frac{1}{x} + \frac{1}{x^2}}{2 + \frac{1}{x} + \frac{5}{x^2}} = \frac{0 + 0}{2 + 0 + 0} = \frac{0}{2} = \boxed{0}$$

② Formulate a rule for what the limit of a rational function $f(x)$ as x approaches either ∞ or $-\infty$ is in the following cases:

(a) If the denominator is of the same power as the numerator.

(b) If the denominator is of a higher power than the numerator.

In case (a), we need only to look at the coefficients of the terms with the highest degree to find its limit. (Look at #1 part A).

$$\lim_{x \to \infty} \frac{\boxed{-4}x^3 - 4x^2 + 3x - 1}{\boxed{9}x^3 + 2x + 5} = \boxed{\frac{-4}{9}}.$$

In case (b), since the denominator is of a higher power, the limit of the rational function will always be <u>ZERO</u> as x approaches either $-\infty$ or ∞.

This is because:

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{P(x)}{q(x)} = \frac{0}{\lim_{x \to \infty} q(x)}.$$

The limit of the numerator as $x \to \infty$ or $-\infty$ goes to zero after it has been divided by x^n, where n is at least a degree higher than the numerator. (Refer to #1 part B).

(3) Let $f(x)$ be defined as

$$f(x) = \frac{x^2+1}{x^2-5x+6}.$$

Find all of the horizontal and vertical asymptotes of $f(x)$ (as well as the right-hand and left-hand limits at each vertical asymptote) and use this to sketch a graph.

Step 1) Determine if the numerator and denominator have common factors that may cancel-out. Suppose they do, and you didn't remove them from your rational function, then you will make a mistake when you try to determine the vertical asymptotes.

$$f(x) = \frac{x^2+1}{x^2-5x+6} = \frac{x^2+1}{(x-3)(x-2)}.$$

There are no common factors in the numerator and denominator that cancel, so we continue...

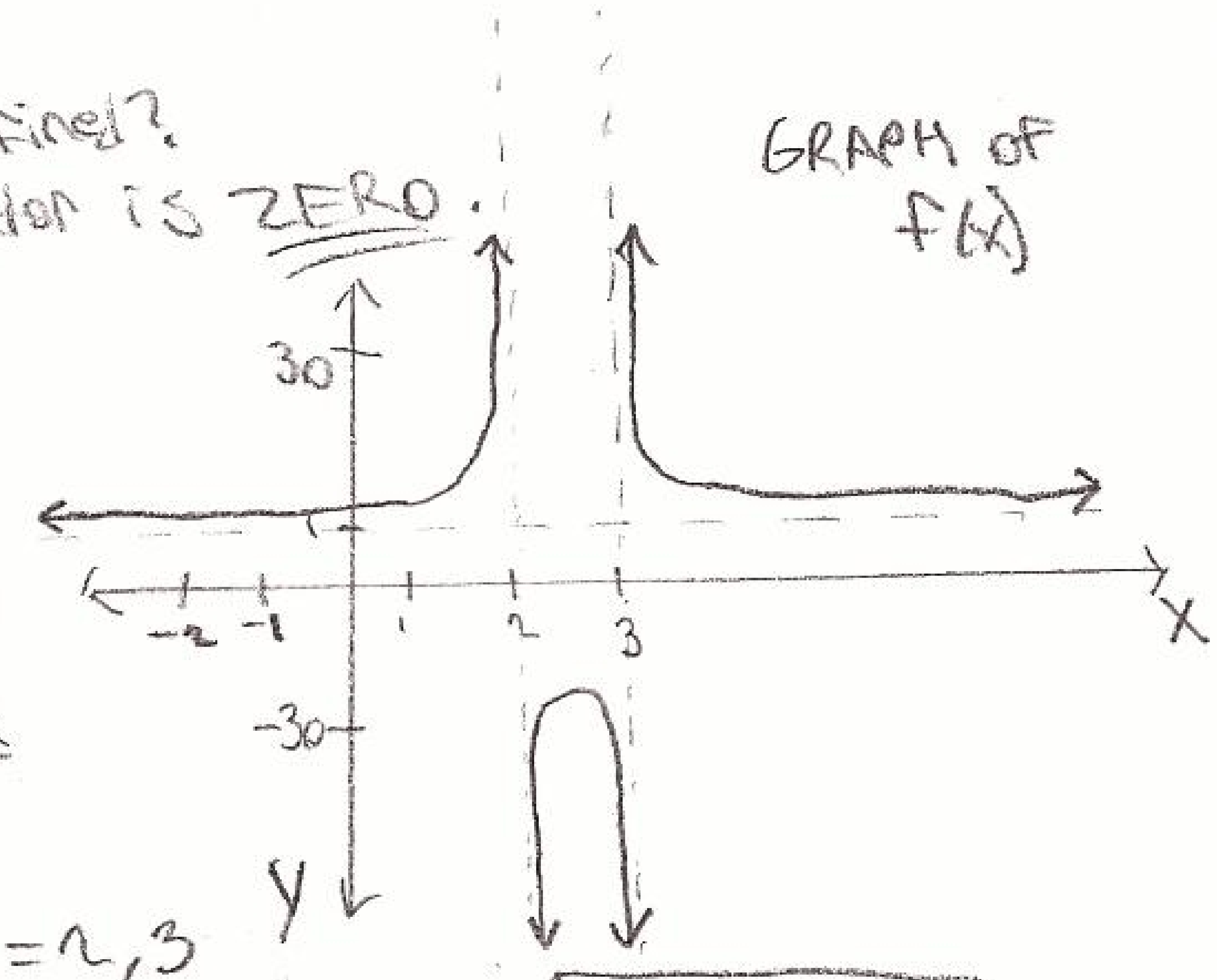
Step 2) Horizontal Asymptotes: Take the limit of $f(x)$ as $x \to \infty$.

$$L = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2+1}{x^2-5x+6} = \frac{1}{1} = \boxed{1}$$

(Degree of top and bottom are the same, so just look at the coefficients on the highest degree term).

Horizontal asymptote at $y=1$.

Step 3) Vertical Asymptotes: When is $f(x)$ not defined?
— When the denominator is ZERO.



Set denominator equal to zero:

$$x^2-5x+6 = (x-3)(x-2) = 0$$

$$\boxed{x = 2, 3}$$

Vertical asymptotes at $x = 2, 3$

Step 4) Check LH (left hand) and RH limits of $x = 2, 3$

(RH) $\lim_{x\to 2^+} \frac{x^2+1}{(x-3)(x-2)} \approx \frac{(2.001)^2+1}{(2.001-3)(2.001-2)} = \frac{4.004001}{(-0.999)(0.001)} = -4008$; $\boxed{\lim_{x\to 2^+} = -\infty}$

(LH) $\lim_{x\to 2^-} \frac{x^2+1}{(x-3)(x-2)} \approx \frac{(1.999)^2+1}{(1.999-3)(1.999-2)} = \frac{4.996001}{(-1.001)(-0.001)} = 4991$; $\boxed{\lim_{x\to 2^-} = \infty}$

(RH) $\lim_{x\to 3^+} \frac{x^2+1}{(x-3)(x-2)} \approx \frac{(3.001)^2+1}{(3.001-3)(3.001-2)} = \frac{10.006001}{(0.001)(1.001)} = 9,996$; $\boxed{\lim_{x\to 3^+} = \infty}$

(LH) $\lim_{x\to 3^-} \frac{x^2+1}{(x-3)(x-2)} \approx \frac{(2.999)^2+1}{(2.999-3)(2.999-2)} = \frac{9.994001}{(-0.001)(0.999)} = -10,004$; $\boxed{\lim_{x\to 3^-} = -\infty}$

Step 5) It's the graph over there

My face when I ran out of paper.