In-Class Work Solutions for March 23rd

Part 1:

1. Consider the curve $y + e^{xy} = x^2$, and say that we're currently at the point $(1, 0)$. Estimate how much the y-coordinate will change if the x-coordinate increases by 0.02.

   Solution:

   Let us find the equation of the tangent line at the point $(1, 0)$. Using implicit differentiation,

$$\begin{aligned}(y + e^{xy})' &= (x^2)' \\ \Rightarrow y' + (xy)'e^{xy} &= 2x \\ \Rightarrow y' + (xy' + y)e^{xy} &= 2x\end{aligned}$$

   To solve, we expand out and then put all the terms with a y' on one side and all the remaining terms on the other side:

$$\begin{aligned}y' + xy'e^{xy} + ye^{xy} &= 2x \\ \Rightarrow y' + xy'e^{xy} &= 2x - ye^{xy} \\ \Rightarrow y'(1 + xe^{xy}) &= 2x - ye^{xy} \\ \Rightarrow y' &= \frac{2x - ye^{xy}}{1 + xe^{xy}}\end{aligned}$$

   Therefore, the tangent line at $(1, 0)$ has the following properties:

$$\begin{aligned}&\text{Point on line} = (1, 0) \\ &\text{Slope of line} = y'(1, 0) = \frac{2 \cdot 1 - 0 \cdot e^0}{1 + 1 \cdot e^0} = \frac{2}{2} = 1\end{aligned}$$

   Using point-slope, the equation of the line is

$$\begin{aligned}y - 0 &= 1 \cdot (x - 1) \\ \Rightarrow y &= x - 1\end{aligned}$$

   Therefore, the linearization at $(1, 0)$ is $L(x) = x - 1$. If the x-coordinate increases by 0.02, it becomes 1.02; therefore, the corresponding y-coordinate will be very close to $L(1.02)$. Thus, we see that

$$y\text{-coordinate at } 1.02 \approx L(1.02) = 1.02 - 1 = 0.02$$

   Since the y-coordinate started off being 0, the increase in y-coordinate is $0.02 - 0 = 0.02$. Therefore, the conclusion is that if the x-coordinate increases by about 0.02,

$$\boxed{\text{The } y\text{-coordinate increases by about } 0.02.}$$

2. Find the linearization of $f(x)$ at $x = 3$ if $f(3) = 4$ and

$$\lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = 2$$

Hint: Don't forget the limit definition of the derivative!

Solution:

By the limit definition of the derivative,

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$$

Therefore, what we're given is that $f'(3) = 2$. To find the equation of the line at $x = 3$, we have that

$$\text{Point on line} = (3, f(3)) = (3, 4)$$
$$\text{Slope of line} = f'(3) = 2$$

Plugging these in, the equation of the line at $(3, f(3))$ is

$$y - 4 = 2(x - 3)$$

Simplifying, $y = 2(x-3)+4 = 2x-6+4 = 2x-2$. Therefore, we conclude that at $x = 3$,

$$\boxed{L(x) = 2x - 2}$$

Part 2:

1. Frank comes home at 1 pm and turns on the heat. The following is a table of values of the temperature $T(t)$ in his house at various times:

t	1 pm	1:15 pm	1:19 pm	1:30 pm
$T(t)$	60°	63°	65°	69°

Estimate the temperature of the house at 1:18 pm.

Solution:

To simplify notation a little, let's say that

$$T(x) = \text{the temperature of Frank's house } x \text{ minutes after 1 pm}$$

That is, $T(0) = 60, T(15) = 63$, etc. Using this notation, we need to estimate $T(18)$. Since we're given $T(19)$, we will use the linearization at $x = 19$. We have that

$$\text{Point on line} = (19, T(19)) = (19, 65)$$
$$\text{Slope of line} = T'(19) = ?$$

We need to estimate the slope of the line at $(19, 65)$. Since we do not have a formula for $T(x)$, we will estimate this using the slope of a secant. The closest x-value to 19 at which we're given the temperature is 15; therefore, we estimate that

$$T'(19) \approx \frac{T(19) - T(15)}{19 - 15} = \frac{65 - 63}{4} = \frac{1}{2}$$

Therefore, using point-slope, we see that the tangent line has approximately the following formula:

$$y - 65 = \frac{1}{2}(x - 19)$$
$$\Rightarrow y = 65 + \frac{1}{2}(x - 19)$$

Thus,

$$L(x) \approx 65 + \frac{1}{2}(x - 19)$$

Now, to estimate $T(18)$, we plug in 18 into the above formula. That is,

$$T(18) \approx L(18) \approx 65 + \frac{1}{2}(18 - 19)$$
$$= 65 - \frac{1}{2} = 64.5$$

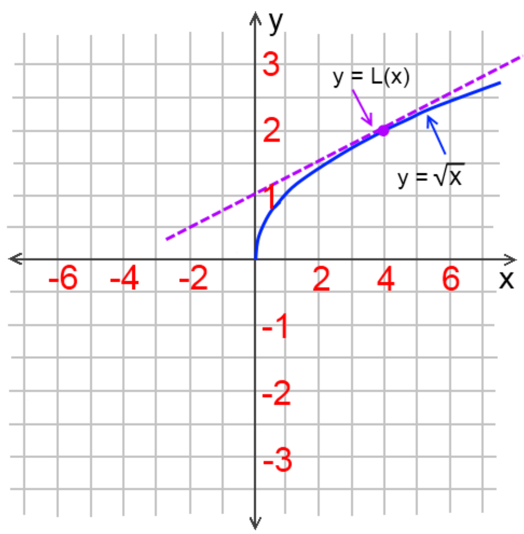
Thus,

$$\boxed{\text{The temperature at } 1:18 \text{ pm is about } 64.5^\circ.}$$

2. Draw a picture to explain whether estimating $\sqrt{3.9}$ with a linearization at $x = 4$ would result in an overestimate or an underestimate. (You do not actually have to do the calculation estimating it.)

Solution:

Let $f(x) = \sqrt{x}$. To estimate $\sqrt{3.9}$, we find the linearization $L(x)$ of $f(x)$ at $x = 4$, and then we plug in 3.9 into $L(x)$. By definition, $L(x)$ is the function whose graph is the the tangent line to $y = f(x)$ at $(4, f(4))$: in this case, it's the tangent line to $y = \sqrt{x}$ at $(4, 2)$. Here's the picture of both $y = \sqrt{x}$ and this tangent line:



It should be clear from the picture that the y-values taken on by $L(x)$ are bigger than the y-values taken on by $\sqrt{x}$ (this is because of the shape of the graph.) Thus, $L(3.9) \geq \sqrt{3.9}$ and therefore

$$\boxed{\text{The estimate will be an overestimate.}}$$