# MATH 408N MIDTERM 1

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02/17/2012
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TA session: ____________________

**Show your work for all the problems. Good luck!**

(1) (a) [5 pts] Write the following expression as a single logarithm. Make sure to simplify as much as possible!

$$3\log_2(3) - \frac{3}{2}\log_2 4 + \log_2\left(\frac{8}{25}\right) + 2\log_2 5$$

**Solution:**

Let us try to simplify all this into a single logarithm. Using log rules, we get

$$\begin{aligned} 3\log_2(3) - \frac{3}{2}\log_2 4 + \log_2\left(\frac{8}{25}\right) + 2\log_2 5 \\ &= \log_2(3^3) - \log_2(4^{3/2}) + \log_2\left(\frac{8}{25}\right) + \log_2(5^2) \\ &= \log_2(27) - \log_2(8) + \log_2\left(\frac{8}{25}\right) + \log_2(25) \end{aligned}$$

using the fact that $4^{3/2} = \left(4^{1/2}\right)^3 = 2^3 = 8$. Continuing, we get

$$\begin{aligned} \log_2\left(\frac{27 \cdot 8/25 \cdot 25}{8}\right) &= \log_2\left(\frac{27 \cdot 8 \cdot 25}{8 \cdot 25}\right) \\ &= \boxed{\log_2(27)} \end{aligned}$$

(b) [5 pts] Let $f(x)$ be defined as

$$f(x) = \frac{2x}{x+3}$$

Find a formula for $f^{-1}(x)$.

**Solution:**

We set up the equation $y = f(x)$, and solve for $x$:

$$\begin{aligned} y &= \frac{2x}{x+3} \\ \Rightarrow (x+3)y &= 2x \\ \Rightarrow xy + 3y &= 2x \\ \Rightarrow 3y &= 2x - xy = x(2-y) \\ \Rightarrow x &= \frac{3y}{2-y} \end{aligned}$$

Now, we swap $x$ and $y$, getting that

$$y = \frac{3x}{2-x}$$

Therefore, $\boxed{f^{-1}(x) = \frac{3x}{2-x}}$.

(2) Let $f(x)$ be the following function:

$$f(x) = \begin{cases} 2x+1 & x<0 \\ 2 & x=0 \\ x^2+x+1 & x>0 \end{cases}$$

(a) [5 pts] Find all values of $a$ for which $\lim_{x\to a} f(x)$ exists. Write your answer in interval notation.

**Solution:**

First note that since the two 'pieces' of $f(x)$ are polynomials, $\lim_{x\to a} f(x)$ exists for all $a \neq 0$. Therefore, we just need to check at $x=0$. Since it's defined separately on the two sides of 0, we consider the left-hand and right-hand limits:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (2x+1) = 2\cdot 0 + 1 = 1$$
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x^2+x+1) = 0^2 + 0 + 1 = 1$$

Since the two limits match, we see that

$$\lim_{x\to 0} f(x) = 1$$

Thus, we see that $\lim_{x\to a} f(x)$ exists for all $a$. Thus, the answer is $\boxed{(-\infty,\infty)}$.

(b) [5 pts] Find all values of $a$ at which the function $f(x)$ is continuous; make sure to use the definition of continuity in your solution for full credit. Write your answer in interval notation.

**Solution:**

Again, since the two pieces are polynomials, $f(x)$ is continuous at all $a \neq 0$. Therefore, we only have to check $a=0$. Using the arguments from part (a), we see that

$$\lim_{x\to 0} f(x) = 1$$

Furthermore, we see from the definition that $f(0)=2$. Thus, we see that

$$\lim_{x\to 0} f(x) \neq 2$$

This means that $f(x)$ isn't continuous at 0. Therefore, $f(x)$ is continuous for all $a$ in $\boxed{(-\infty,0)\cup(0,\infty)}$.

(3) Let $f(x) = \dfrac{x^2 - 1}{x^2 - 3x + 2}$.

(a) [5 pts] Find all the horizontal asymptotes of $f(x)$. Make sure to show all your work.

**Solution:**

To find the horizontal asymptotes of $f(x)$, we take the limit of $f(x)$ as $x$ approaches $\infty$ and $-\infty$. The way to do this is to divide top and bottom by the highest power of $x$ in the denominator:

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{x^2-1}{x^2-3x+2} = \lim_{x\to\infty} \frac{\frac{1}{x^2}(x^2-1)}{\frac{1}{x^2}(x^2-3x+2)} \\ &= \lim_{x\to\infty} \frac{1-\frac{1}{x^2}}{1-\frac{3}{x}+\frac{2}{x^2}} = \frac{1-0}{1-0+0} = 1\end{aligned}$$

Similarly,

$$\begin{aligned}\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{x^2-1}{x^2-3x+2} = \lim_{x\to\infty} \frac{1-\frac{1}{x^2}}{1-\frac{3}{x}+\frac{2}{x^2}} \\ &= \frac{1-0}{1-0+0} = 1\end{aligned}$$

Therefore, the only horizontal asymptote is $\boxed{y = 1}$. (Note that you do have to check both $\infty$ and $-\infty$: if the two limits were different, then we'd have two different asymptotes!)

(b) [5 pts] Find all the vertical asymptotes of $f(x)$. Make sure to show all your work.

**Solution:**

Recall that vertical asymptotes occur when either the denominator is 0 or the numerator blows up. Furthermore, if both denominator and numerator are 0, we have to do more checking. Here, the numerator never blows up. Thus, set denominator to 0:

$$\begin{aligned} x^2 - 3x + 2 &= 0 \\ \Rightarrow (x-2)(x-1) &= 0\end{aligned}$$

Thus, the only possibilities are $x = 1$ and $x = 2$. When $x = 2$, the numerator is $2^2 - 1 = 3$, so $x = 2$ is indeed an asymptote. When $x = 1$, the numerator is $1^2 - 1 = 0$, so we need to check. We see that

$$\begin{aligned}\lim_{x\to 1} f(x) &= \lim_{x\to 1} \frac{x^2-1}{x^2-3x+2} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(x-2)} \\ &= \lim_{x\to 1} \frac{x+1}{x-2} = \frac{2}{-1} = -2\end{aligned}$$

Since the limit at 1 is a number, that means that the function is not going off to $\infty$ around $x = 1$. This means $x = 1$ isn't an asymptote. Therefore, the only vertical asymptote is $\boxed{x = 2}$.

(4) Do the following questions:

(a) [5 pts] Calculate the following limit:

$$\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right)$$

Show every step in your work. Justify any calculation that requires it.

**Solution:**

Here, we first simplify the expression as a single fraction, and then try to evaluate the limit. (Trying to plug in right now results in expressions like $\infty - \infty$, which are completely meaningless.)
Working it out,

$$\begin{aligned}\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right) &= \lim_{x\to 1}\left(\frac{x+1}{(x-1)(x+1)}-\frac{2}{x^2-1}\right)\\ &= \lim_{x\to 1}\left(\frac{x+1}{x^2-1}-\frac{2}{x^2-1}\right)\\ &= \lim_{x\to 1}\left(\frac{x+1-2}{x^2-1}\right) = \lim_{x\to 1}\left(\frac{x-1}{(x-1)(x+1)}\right)\\ &= \lim_{x\to 1}\left(\frac{1}{x+1}\right) = \boxed{\frac{1}{2}}\end{aligned}$$

(b) [5 pts] Use the rules learned in class so far to calculate the derivative of the following function:

$$f(x) = \frac{x-\sqrt[3]{x}}{x^2}+\frac{4}{\sqrt{x}}+\pi-2e^{x-1}$$

**Solution:**

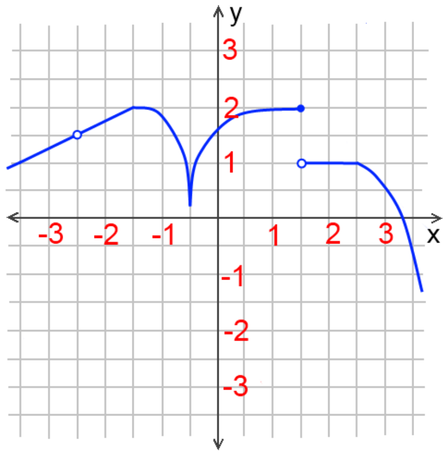
Here, we need to rewrite everything as a sum of powers of $x$ and exponential functions. Accordingly,

$$\begin{aligned}f(x) &= \frac{x}{x^2}-\frac{\sqrt[3]{x}}{x^2}+\frac{4}{\sqrt{x}}+\pi-2e^{x-1}\\ &= x^{-1}-\frac{x^{1/3}}{x^2}+4x^{-1/2}+\pi-2e^{-1}e^{x}\\ &= x^{-1}-x^{-5/3}+4x^{-1/2}+\pi-2e^{-1}e^{x}\end{aligned}$$

Therefore, differentiating:

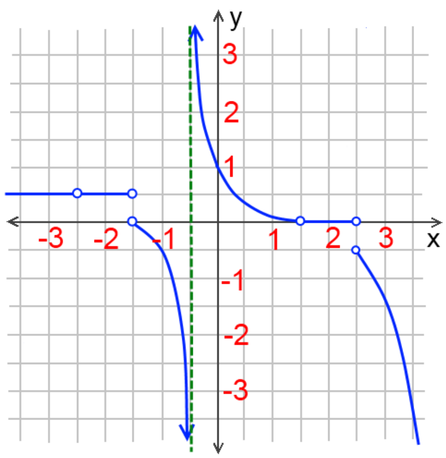
$$\begin{aligned}f'(x) &= (x^{-1}-x^{-5/3}+4x^{-1/2}+\pi-2e^{-1}e^{x})'\\ &= (x^{-1})'-(x^{-5/3})'+4(x^{-1/2})'-2e^{-1}(e^{x})'\\ &= x^{-2}-\left(-\frac{5}{3}\right)x^{-8/3}+4\cdot\left(-\frac{1}{2}\right)x^{-3/2}-2e^{-1}e^{x}\\ &= \boxed{x^{-2}+\frac{5}{3}x^{-8/3}-2x^{-3/2}-2e^{x-1}}\end{aligned}$$

(5) [10 pts] For the $f(x)$ in the following picture, graph $f'(x)$ on the empty axes below. Make sure to estimate the values of $f'(x)$ carefully, and also to record whether $f'(x)$ is increasing or decreasing on the graph. Also, write a couple of sentences about how you graphed what's going on at $x = -0.5$.




**Solution:**

Here is the picture of $f'(x)$:




To figure out what's going on at $x = -0.5$, we need to see what's happening to the derivative as $x$ approaches $-0.5$ from the left, and $x$ approaches $-0.5$ from the right. We see that to the left of $-0.5$, the tangent lines to the graph are getting arbitrarily steep, and since the slopes are negative, $f'(x)$ is approaching $-\infty$. To the right of $-0.5$, the tangent lines are also getting arbitrarily steep; however, since the slopes are positive, $f'(x)$ is approaching $\infty$. This explains the presence of the asymptote $x = -0.5$ on the graph, as well as the behaviour of $f'(x)$ near the asymptote.

(6) [7 pts] BONUS: Explain where the following equation comes from:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

For full points on this question, you must both provide a good picture, and a clear explanation of what's happening!

**Solution:**

Coming soon!