MATH 408N MIDTERM 2

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TA session: ____________________

Show your work for all the problems. Good luck!

(1) (a) [5 pts] Use the limit definition of the derivative to calculate $f'(x)$, if $f(x) = x^2$.

Solution:

By definition,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0} (2x+h) \\
&= \boxed{2x}
\end{aligned}$$

(b) [5 pts] Find $f''(3)$ using whatever rules you like, if

$$f(x) = 3g(x)^2$$

and we're given that $g(3) = 1, g'(3) = 2$ and $g''(3) = -1$.

Solution:

Using the chain rule,

$$f'(x) = 3 \cdot 2g(x)g'(x) = 6g(x)g'(x)$$

Now, using the product rule,

$$\begin{aligned}
f''(x) &= (f'(x))' = (6g(x)g'(x))' \\
&= 6(g(x)(g'(x))' + g'(x)g'(x)) \\
&= 6(g(x)g''(x) + g'(x)^2)
\end{aligned}$$

Thus, plugging in $x = 3$,

$$\begin{aligned}
f''(3) &= 6(g(3)g''(3) + g'(3)^2) = 6(1 \cdot (-1) + 2^2) = 6 \cdot 3 \\
&= \boxed{18}
\end{aligned}$$

(2) Calcuate the derivatives of the following functions, using whatever rules you like. You do *not* need to simplify your answer, but it should be written **only** in terms of x!

(a) [5 pts] $f(x) = \dfrac{\sin(x)}{x^2-1} + \dfrac{\cos^{-1}(x)}{6} + 3^{x+1}$.

Solution:

Since this is a sum of functions, it suffices to differentiate each of the functions separately. Using the quotient rule,

$$\begin{aligned}\left(\frac{\sin(x)}{x^2-1}\right)' &= \frac{(x^2-1)(\sin(x))' - (x^2-1)'\sin(x)}{(x^2-1)^2}\\ &= \frac{(x^2-1)\cos(x) - 2x\sin(x)}{(x^2-1)^2}\end{aligned}$$

Differentiating the second term,

$$\begin{aligned}\left(\frac{\cos^{-1}(x)}{6}\right)' &= \left(\frac{1}{6}\cos^{-1}(x)\right)' = \frac{1}{6}(\cos^{-1}(x))'\\ &= \frac{1}{6}\cdot\left(-\frac{1}{\sqrt{1-x^2}}\right) = -\frac{1}{6\sqrt{1-x^2}}\end{aligned}$$

Finally, using the chain rule:

$$(3^{x+1})' = 3^{x+1}\ln(3)(x+1)' = 3^{x+1}\ln(3)$$

Thus, putting it all together,

$$\boxed{f'(x) = \frac{(x^2-1)\cos(x) - 2x\sin(x)}{(x^2-1)^2} - \frac{1}{6\sqrt{1-x^2}} + 3^{x+1}\ln(3)}$$

(b) [5 pts] $f(x) = (x^2+1)^{\tan(x)}$.

Solution:

Here, we have to use logarithmic differentiation:

$$\begin{aligned} y &= (x^2+1)^{\tan(x)}\\ \Rightarrow \ln(y) &= \ln\left((x^2+1)^{\tan(x)}\right) = \tan(x)\ln(x^2+1)\end{aligned}$$

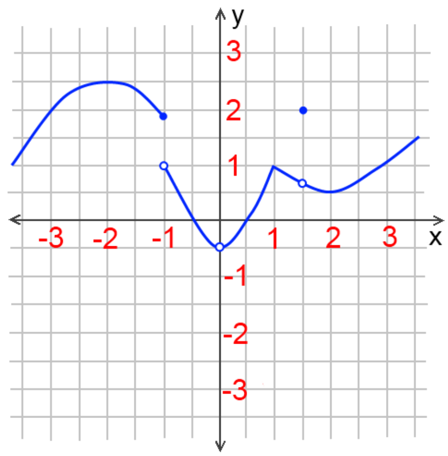
Differentiating both sides with respect to x,

$$\begin{aligned}\frac{y'}{y} &= (\tan(x))'\ln(x^2+1) + \tan(x)(\ln(x^2+1))'\\ &= \sec^2(x)\ln(x^2+1) + \tan(x)\frac{1}{x^2+1}(x^2)'\\ &= \sec^2(x)\ln(x^2+1) + \frac{2x\tan(x)}{x^2+1}\\ \Rightarrow y' &= y\left(\sec^2(x)\ln(x^2+1) + \frac{2x\tan(x)}{x^2+1}\right)\end{aligned}$$

Finally, plugging in the expression for y, we get

$$\boxed{f'(x) = (x^2+1)^{\tan(x)}\left(\sec^2(x)\ln(x^2+1) + \frac{2x\tan(x)}{x^2+1}\right)}$$

(3) Let $f(x)$ be given in the following picture:



(a) [5 pts] Find all x-coordinates at which the local minimums and maximums are attained; specify whether each one is a min or a max.

Solution:

By definition, a local minimum is a value a such that $f(a) \leq f(x)$ for all x sufficiently close to a. A similar definition holds for local maximum. Therefore,

A local minimum is attained at $x = 2$.

Local maximums are attained at $x = -2, 1, 1.5$.

(b) [5 pts] Find the absolute minimum and the absolute maximum of $f(x)$ on $[-3, 3]$, and the x-coordinates at which they are attained. If either of these doesn't exist, justify why not.

Solution:

The absolute minimum is the largest value attained by $f(x)$ on an interval. For this function,

The absolute maximum is 2.5, attained at $x = -2$.

There is no absolute minimum.

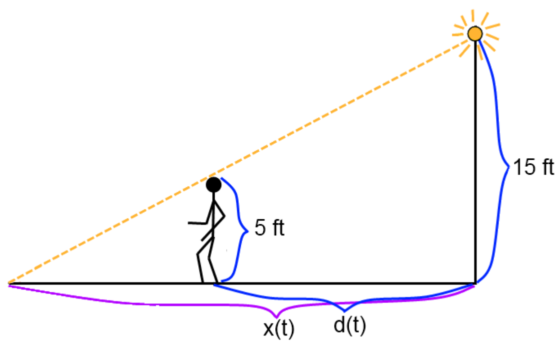
There is no absolute minimum because there is no smallest value attained by $f(x)$ – for any y-value that is attained, we can get a smaller one by getting sufficiently close to $x = 0$. (This is because it looks like the smallest y-value is at the "open circle" $(0, -0.5)$.)

(4) [10 pts] There is a streetlight on top of a 15 foot pole. A person who is 5 foot tall walks away from the pole at 2 feet per second. How quickly is the tip of his shadow moving? Please write down your final answer as an English sentence, and include whether it's moving towards or away from the pole.

Note: Make sure to set up your problem so that you're measuring what you're being asked about!

Solution:

1. Here's the diagram.



2. Here, $d(t)$ is the distance of the man from the pole, and $x(t)$ is the distance of the tip of his shadow from the pole.
3. Given: $d'(t) = 2$.
4. Need to find: $x'(t)$. This will be the velocity of the tip of the shadow, since $x(t)$ is the position of the tip with respect to the base of the streetlight.
5. Relationship: As usual in these problem, we have similar triangles. The triangle formed by the man and his shadow is similar to the triangle formed by the streetlight and the distance of the tip of the shadow to the pole. The way the variables are labelled, the length of the man's shadow is precisely $x(t) - d(t)$. Thus, the relationship is
$$\frac{x(t) - d(t)}{5} = \frac{x(t)}{15}$$
Multiplying both sides by 15 and simplifying, this becomes,
$$\begin{aligned} \frac{15}{5}(x(t) - d(t)) &= x(t) \\ \Rightarrow 3x(t) - 3d(t) &= x(t) \\ \Rightarrow 2x(t) &= 3d(t) \\ \Rightarrow x(t) &= \frac{3}{2}d(t) \end{aligned}$$
6. Differentiate both sides:
$$x'(t) = \frac{3}{2}d'(t)$$
7. Substitute information given: we're given that $d'(t) = 2$, and therefore
$$x'(t) = \frac{3}{2} \cdot 2 = 3$$
Therefore,

$\boxed{\text{The tip of the person's shadow is moving away from the pole at 3 feet per second.}}$

(5) Consider the curve given by the equation:

$$2\sin(y^2-1)+e^{x+y}=x^2$$

(a) [5 pts] Find the equation of the tangent line to this curve at the point $(-1,1)$.

Solution:

Differentiating both sides with respect to x (not forgetting to use implicit differentiation!), we get

$$\begin{aligned} 2\cos(y^2-1)\cdot 2y\cdot y'+e^{x+y}(x+y)' &= 2x \\ \Rightarrow 4yy'\cos(y^2-1)+(1+y')e^{x+y} &= 2x \\ \Rightarrow 4yy'\cos(y^2-1)+e^{x+y}+y'e^{x+y} &= 2x \end{aligned}$$

Moving all terms with a y' to one side and then factoring out the y', we get

$$\begin{aligned} 4yy'\cos(y^2-1)+y'e^{x+y} &= 2x-e^{x+y} \\ \Rightarrow y'(4y\cos(y^2-1)+e^{x+y}) &= 2x-e^{x+y} \end{aligned}$$

Finally, dividing both sides by $4y\cos(y^2-1)+e^{x+y}$, we get

$$y'=\frac{2x-e^{x+y}}{4y\cos(y^2-1)+e^{x+y}}$$

Since we're looking for the tangent line at $(-1,1)$, we need the slope at $(-1,1)$. Thus, plugging in, we see that

$$\begin{aligned} \text{Slope}=y'(-1,1) &= \frac{2\cdot(-1)-e^{(-1)+1}}{4\cdot 1\cdot\cos(1^2-1)+e^{(-1)+1}} \\ &= \frac{-2-1}{4+1}=-\frac{3}{5} \end{aligned}$$

Thus, using the point slope formula, the equation of the tangent line is:

$$(y-1)=-\frac{3}{5}(x-(-1))$$

Simplifying, this is just

$$\boxed{y=-\frac{3}{5}(x+1)+1=-\frac{3}{5}x+\frac{2}{5}}$$

(b) [5 pts] Use the result from part (a) to estimate how much the y-coordinate would change, if we started at the point $(-1,1)$ on our curve and decreased the x-coordinate by 0.02. Please write your answer as a fraction, and state whether the y-coordinate would increase or decrease.

Solution:

Decreasing the x-coordinate by 0.02 means that instead of having the coordinate -1, we have the x-coordinate -1.02. To estimate the y-coordinate at this point, we plug this x-coordinate into the equation of the tangent line:

$$\begin{aligned} y &\approx -\frac{3}{5}(-1.02+1)+1=-\frac{3}{5}\cdot(-0.02)+1 \\ &= \frac{3}{5}\frac{2}{100}+1=1+\frac{6}{500}=1+\frac{3}{250} \end{aligned}$$

Thus, the y-coordinate starts at 1, and changes to $1+3/250$. Therefore,

$$\boxed{\text{The } y\text{-coordinate increases by approximately } \frac{3}{250}.}$$

(6) [5 pts] BONUS: Figure out the derivative of $\log_2(x)$ using the fact that $\log_2(x)$ is the inverse of a function whose derivative you know.

Solution:

Since $\log_2(x)$ is the inverse of 2^x, letting $y = \log_2(x)$,

$$\begin{aligned} y &= \log_2(x) \\ \Rightarrow 2^y &= x \end{aligned}$$

Now, differentiating both sides with respect to x,

$$\begin{aligned} 2^y \ln(2) y' &= 1 \\ \Rightarrow y' &= \frac{1}{\ln(2)2^y} \end{aligned}$$

Finally, plug in $y = \log_2(x)$ to find y' in terms of x:

$$y' = \frac{1}{\ln(2)2^{\log_2(x)}} = \frac{1}{\ln(2)x}$$

Therefore,

$$\boxed{(\log_2(x))' = \frac{1}{\ln(2)x}}$$