MATH 408N PRACTICE MIDTERM 1

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TA session: ____________________

Show your work for all the problems. Good luck!

(1) (a) [5 pts] Solve for x if

$$2^{x+3} = 4^{3x-1}$$

Solution:

Writing everything as a power of 2,

$$2^{x+3} = (2^2)^{3x-1} = 2^{2(3x-1)} = 2^{6x-2}$$

using exponent rules and expanding things out.
To have powers of the same base be equal, the exponents have to be the same. Therefore,

$$\begin{aligned} x + 3 &= 6x - 2 \\ \Rightarrow 5 &= 5x \\ \Rightarrow x &= 1 \end{aligned}$$

Thus, the answer is $\boxed{x = 1.}$

(b) [10 pts] Let

$$f(x) = \frac{e^x}{e^x + 1}$$

Find a formula for $f^{-1}(x)$, and make your answer as simple as possible by using logarithm rules.

Solution:

Start by rewriting the equation as

$$y = \frac{e^x}{e^x + 1}$$

Now, we need to solve for x in terms of y. We first solve for e^x. Beging by multiplying both sides by the denominator $e^x + 1$:

$$\begin{aligned} (e^x + 1) \times y &= \frac{e^x}{e^x + 1} \times (e^x + 1) \\ \Rightarrow (e^x + 1)y &= e^x \\ \Rightarrow e^x y + y &= e^x \\ \Rightarrow e^x y - e^x &= -y \\ \Rightarrow e^x(y - 1) &= -y \\ \Rightarrow e^x &= \frac{-y}{y - 1} \end{aligned}$$

Now, taking ln of both sides we get

$$x = \ln(e^x) = \ln\left(\frac{-y}{y-1}\right) = \ln(-y) - \ln(y - 1)$$

Switching x and y, we see that the answer is $\boxed{f^{-1}(x) = \ln(-x) - \ln(x - 1)}$. (If you made a different choice when solving for e^x, you would get $\ln(x) - \ln(1 - x)$, which is the same thing.)

(2) [10 pts] Let $f(x)$ be defined as follows:

$$f(x) = \begin{cases} x & x \leq 0 \\ x^2 & 0 < x < 1 \\ 1 - x & 1 \leq x \end{cases}$$

Which values of a is this function continuous at? State your answer in interval notation. Make sure to show all the appropriate limit calculations and justify continuity for all stated values of a!

Solution:

As noted in class, I recommend starting this question by drawing a picture. While I will not do so in this soluion, you will the logic easier to follow if you sketch your own picture before reading it.

$f(x)$ is a piecewise function with three different "pieces." Each of this pieces is a polynomial: as a result, $f(x)$ is definitely continuous everywhere except where those pieces "connect." Thus, we only need to check whether $f(x)$ is continuous at 0 and 1.

Checking $x = 0$: By definition, $f(x)$ is continous at 0 if and only if

$$f(0) = \lim_{x \to 0} f(x)$$

This means that we need to check two things:
(a) Does $\lim_{x \to 0} f(x)$ exist?
(b) If the limit exists, does it equal to $f(0)$?

To check whether the limit exists, we check whether the right-hand and left-hand limit match. As is clear from the piecewise definition (and should be extra clear from the picture), $f(x)$ is defined to be x a little to the left of 0, and is defined to be x^2 a little to the right of 0. Therefore,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x = 0$$
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} x^2 = 0^2 = 0$$

where we use direct substitution for the limits, as they are limits of polynomials. Therefore, we see that $\lim_{x \to 0} f(x)$ exists and is equal to 0. By the definition of f, $f(0) = 0$. Thus, we see that

$$f(0) = \lim_{x \to 0} f(x)$$

and therefore $f(x)$ is continuous at 0.

Checking $x = 1$: Similiarly to above, we need to check whether

$$f(1) = \lim_{x \to 1} f(x)$$

Again, break this up into checking whether the limit exists, and if it does, whether it's equal to $f(1)$. We have that

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1$$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (1 - x) = 0$$

Thus, the right-hand and left-hand limits don't match, and therefore the limit doesn't exist. This means that $f(x)$ is not continuous at 1.

The above calculations show that $f(x)$ is continuous everywher but at $x = 1$. Therefore,

$$\boxed{f(x) \text{ is continuous on } (-\infty, 1) \cup (1, \infty)}$$

(3) Calculate the following limits. You must show all your work to get credit. State if you're using continuity.

(a) [5 pts] $\lim_{x\to 0} \dfrac{\sqrt{3x+4}-2}{x}$

Solution:

Here, direct substitution results in $\frac{0}{0}$, which means that $x = 0$ is not in the domain of the function. Therefore, we need to do some simplifying calculations – we use the difference of squares formula after multiplying both top and bottom by the conjugate of the top:

$$\begin{aligned}\lim_{x\to 0} \frac{\sqrt{3x+4}-2}{x} &= \lim_{x\to 0} \frac{\sqrt{3x+4}-2}{x} \cdot \frac{\sqrt{3x+4}+2}{\sqrt{3x+4}+2} \\ &= \lim_{x\to 0} \frac{(\sqrt{3x+4})^2-2^2}{x(\sqrt{3x+4}+2)} = \lim_{x\to 0} \frac{3x+4-4}{x(\sqrt{3x+4}+2)} \\ &= \lim_{x\to 0} \frac{3x}{x(\sqrt{3x+4}+2)} \\ &= \lim_{x\to 0} \frac{3}{\sqrt{3x+4}+2}\end{aligned}$$

canceling out the x in the top and bottom in the last step. We're now at the point where we can do direct substitution, since the function $f(x) = \frac{3}{\sqrt{3x+4}+2}$ is continuous on its domain, and $x = 0$ is in its domain. Therefore,

$$\lim_{x\to 0} \frac{3}{\sqrt{3x+4}+2} = \lim_{x\to 0} \frac{3}{\sqrt{3\cdot 0+4}+2} = \frac{3}{\sqrt{4}+2} = \frac{3}{4}$$

and therefore,

$$\boxed{\lim_{x\to 0} \frac{\sqrt{3x+4}-2}{x} = \frac{3}{4}}$$

(b) [5 pts] $\lim_{x\to\infty} \dfrac{x^2+x+1}{2x^2-x+3}$

Solution:

Start by diving both top and bottom by the highest power of x in the denominator, which happens to be x^2:

$$\begin{aligned}\lim_{x\to\infty} \frac{x^2+x+1}{2x^2-x+3} &= \lim_{x\to\infty} \frac{(x^2+x+1)/x^2}{(2x^2-x+3)/x^2} \\ &= \lim_{x\to\infty} \frac{1+\frac{1}{x}+\frac{1}{x^2}}{2-\frac{1}{x}+\frac{3}{x^2}} \\ &= \frac{\lim_{x\to\infty}(1+\frac{1}{x}+\frac{1}{x^2})}{\lim_{x\to\infty}(2-\frac{1}{x}+\frac{3}{x^2})} \\ &= \frac{\lim_{x\to\infty} 1+\lim_{x\to\infty}\frac{1}{x}+\lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty} 2-\lim_{x\to\infty}\frac{1}{x}+\lim_{x\to\infty}\frac{3}{x^2}} = \boxed{\frac{1}{2}}\end{aligned}$$

using the fact that for any $r > 0$, $\frac{1}{x^r}$ approaches 0 as x approaches ∞.

(c) [5 pts] $\lim_{x\to 1^-} \dfrac{x+1}{x^2-3+2}$
Hint: You might want to factor the denominator first...

Solution:

Note that direct substitution yields $\frac{2}{0}$. This means that direct substitution doesn't work, and that the answer is probably going to be either ∞ or $-\infty$. We just need to decide which one.
As noted in the hint, start by factoring the denominator:

$$\lim_{x\to 1^-} \frac{x+1}{x^2-3+2} = \lim_{x\to 1^-} \frac{x+1}{(x-1)(x-2)}$$

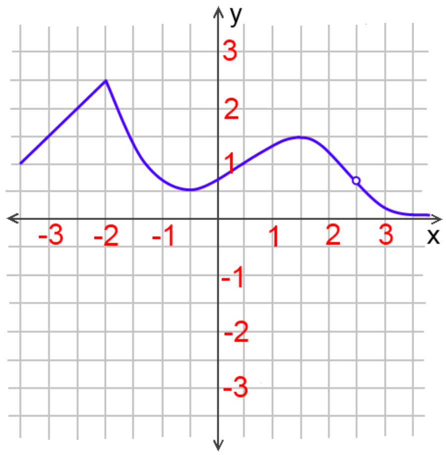
To see whether what the function is approaching as $x \to 1^-$, plug in a number a little to the left of 1, like 0.999:

$$\begin{aligned}\lim_{x\to 1^-} \frac{x+1}{(x-1)(x-2)} &\approx \frac{1+0.999}{(0.999-1)(0.999-2)} \\ &\approx \frac{2}{(\text{Small negative number})\cdot(-1)} \\ &= \frac{2}{\text{Small positive number}} = \text{Big positive number}\end{aligned}$$

Therefore,

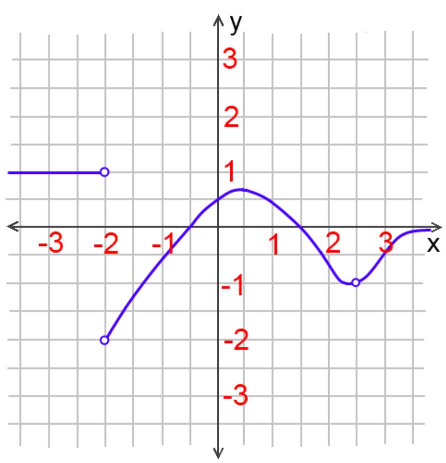
$$\boxed{\lim_{x\to 1^-} \frac{x+1}{x^2-3x+2} = \infty}$$

(4) (a) [10 pts] Let $f(x)$ be given in the following graph. Sketch the graph of $f'(x)$ on the empty axes below. Make sure to estimate the values of $f'(x)$ carefully, and also to record whether $f'(x)$ is increasing or decreasing on the graph.



Solution:
Before sketching, we note the following features:

- The slope (derivative) is constant and equal to 1 for $x \leq -2$.
- The derivative is not defined at -2.
- The derivative is negative (starting about about -2) to the right of 2, increases until it's 0 at $x = -0.5$, keeps increasing until about 0.7 at $x = 0.5$, then again decreases until it's 0 at $x = 1.5$.
- Finally, the derivative becomes negative again, decreasing until about -1 a bit to the right of 2, then increasing and becoming close to the x-axis.
- The derivative is not defined at $x = 2.5$, since $f(x)$ is not continuous there.



(b) [5 pts] Find $f'(x)$, if

$$f(x) = \frac{x^2 - 2x}{3x^3} + \frac{1}{2\sqrt{x}} + e^{x-1}$$

Use only the rules we have learned in class so far.

Solution:

The trick is to write $f(x)$ as a sum of powers of x (and e^x) times constants. Simplifying, we see that

$$\begin{aligned} f(x) &= \frac{x^2}{3x^3} - \frac{2x}{3x^3} + \frac{1}{2}\frac{1}{\sqrt{x}} + e^{-1}e^x \\ &= \frac{1}{3}x^{-1} - \frac{2}{3}x^{-2} + \frac{1}{2}x^{-1/2} + e^{-1}e^x \end{aligned}$$

Thus, using differentiation rules, we see that

$$\begin{aligned} f'(x) &= \frac{1}{3}\cdot(-1)x^{-2} - \frac{2}{3}\cdot(-2)x^{-3} + \frac{1}{2}\cdot\left(-\frac{1}{2}\right)x^{-3/2} + e^{-1}e^x \\ &= \boxed{-\frac{1}{3}x^{-2} + \frac{4}{3}x^{-3} - \frac{1}{4}x^{-3/2} + e^{x-1}} \end{aligned}$$